

A Unified CRT Framework for Zero-Divisor Graphs over General Composite Integers: Structural Reduction and Invariant Transfer

Kannan M¹, Sasikala A²

¹Research Scholar, Department of Mathematics, Periyar Maniammai Institute of Science & Technology (Deemed to be University), Vallam, Thanjavur - 613403, Tamil Nadu, India, E-mail: kannanlathatrichy@gmail.com

²Professor, Department of Mathematics, Periyar Maniammai Institute of Science & Technology (Deemed to be University), Vallam, Thanjavur - 613403, Tamil Nadu, India, E-mail: sasikala@pmu.edu

HOW TO CITE:

Kannan M, Sasikala A (2026). A Unified CRT Framework for Zero-Divisor Graphs over General Composite Integers: Structural Reduction and Invariant Transfer. International Journal of Special Education, 41(22s), 1539-1549

COPYRIGHT STATEMENT:

Copyright: © 2026 Authors.
Open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).

ABSTRACT:

The results on the zero-divisor graphs of modular rings are often given only for prime powers, semiprimes or products of two prime powers, and there is no single computational and theoretical model for an arbitrary composite integer. This paper builds one of these in the case of $\Gamma(\mathbb{Z}_n)$, where $n = \prod p_i^{a_i}$ is any product of distinct prime numbers and arbitrary positive exponents. Each nonzero zero-divisor is encoded by its signature, which is a set of signatures $\alpha = (\min(v_{p_i}(x), a_i))_i$. Each nonzero zero-divisor is coded by its signature $\alpha = (\min(v_{p_i}(x), a_i))_i$. Every class size has a multiplicative formula according to the Chinese remainder theorem and the zero product adjacency is the coordinate wise condition $\alpha_i + \beta_i \geq a_i$. This results in a reconstruction of $\Gamma(\mathbb{Z}_n)$ to be exactly a weighted quotient of a clique or independent set with $\tau(n)-2$ classes. Adjacency and Laplacian spectra, pairwise distances and the Wiener index, domination, metric dimension bounds and demand-preserving coloring are then considered in general transfer results. The construction is not affected by the number or multiplicities of prime factors. No adjacency or spectral mismatch have been found after exhaustive deterministic verification of all 138 composite integers $4 \leq n \leq 180$. The mean structural reduction of the quotient was 85.6% and the reductions of nontrivial cases ($n \geq 6$) varied from 33.3% to 97.8%. Ten moduli are shown which are representative examples of the effect on the transferred invariants of different factorization patterns. The main result is a family-independent reproducible CRT engine rather than a number of individual, family-specific derivations.

Keywords: zero-divisor graph, Chinese remainder theorem, General composite integer, Valuation signature, Weighted quotient.

1. Introduction

Graphs on algebraic systems represent multiplication, annihilation and ideal interaction as adjacency. This direction was started by Beck's coloring graph [9] and is continued in Anderson and Livingston's zero-divisor graph [5]. Let R be a commutative ring. The vertices correspond to the nonzero zero-divisors of R , and the vertices

corresponding to nonzero zero-divisors x, y of R are adjacent if and only if $xy = 0$. If the graph is not empty, the order and organization of the graph may vary, but the graph is connected with diameter at most three when using the ring [3, 6, 18]. Planarity, multipartite structure, girth and extensions to semigroups have thus been studied in various ways [2, 14, 29].

For $R=\mathbb{Z}_n$ the prime factorization provides extra "algebraic" data. However, the vast majority of published work has been done family by family: prime powers, semiprimes, or $p^\alpha q^\beta$ and other carefully chosen notations and derivations are done on each family separately [8, 12, 17, 19, 23, 24, 28]. This specialization may make people forget that all the n are composites with the same coordinate mechanism (CRT). It also gives the impression that extension to 3 or more prime factors is like constructing a new building when really it is still a case of multiplying to zero, coordinate by coordinate.

A beginning is provided by compression by equal annihilators. In $\mathbb{Z}_n$, equal annihilators have equal gcd, and repeated residues in graphs are replaced by annihilators of the graph in $\mathbb{Z}_n$, which are divisor classes of [1, 4, 31]. However, a scalar divisor label does not indicate which prime-power coordinate is responsible for compatibility; an unweighted compressed graph, on the other hand, does not even remember class multiplicity or even what is internal and what is not. A truly general framework needs to work directly with an full exponent vector $a=(a_1, \dots, a_r)$, be able to take any r , keep class sizes, and generate internal and cross-class edges without relying on case distinctions.

The lack of a unified framework for a general composite integer $n=\prod p_i^{a_i}$, without any

restrictions on the number of different prime factors or on their powers, using a CRT-based method, is the research gap. The aim is to create a universal valuation-signature quotient and to demonstrate that it is enough to recover $\Gamma(\mathbb{Z}_n)$ and to pass major invariants. Rather than as unrelated goals, the sufficiency tests of Spectrum, Distance, Domination, Metric Dimension and Coloring are challenging. The contributions include a factorization independent class formula, a coordinate wise compatibility theorem, lossless generalized-join reconstruction, class invariant transfer results and exhaustive verification of all composites, $n \leq 180$.

2 Preliminaries and notation

Let the composite number $n \geq 4$ have the factorization $n = \prod_{i=1}^r p_i^{a_i}$, where p_i are prime numbers. $G_n = \Gamma(\mathbb{Z}_n)$ is a graph with vertices x and y if x is in the set $Z^*(\mathbb{Z}_n) = \{x \in \{1, 2, \dots, n-1\} : \gcd(x, n) > 1\}$ then x and y are adjacent if $xy \equiv 0 \pmod{n}$ [5]. For the adjacency matrix $A(G)$ find the Laplacian matrix $L(G)$ and the spectral radius $\rho(G)$ of an undirected graph G , algebraic connectivity $\mu_2(G)$ of an undirected graph G , chromatic number $\chi(G)$ of an undirected graph G , domination number $\text{dom}(G)$ of an undirected graph G , metric dimension of an undirected graph G and its Wiener index $W(G)$. The standard spectral and metric notation is followed, see [11, 15, 16].

Table 1: Principal notation

Symbol	Definition or role
$a=(a_1, \dots, a_r)$	prime-power exponent vector of n
$\alpha(x)$	truncated valuation signature of residue x
C_α, s_α	signature class and its cardinality
ϵ_α	1 for a clique class, 0 for an independent class
Q_n	compatibility quotient on nontrivial signatures
B_A, B_L	symmetric reduced adjacency and Laplacian matrices
d_Q	shortest-path distance in the quotient
t_α, z_α	class count and activation variables in domination

A set S is a dominating set of a graph, if all vertices of the graph are either in S or have a neighbour in S [13]. The smallest size of a resolving set is the metric dimension [11, 16]. The concepts of domination and metric dimension

should not be confused: domination is about coverage, while metric dimension is about identification. Both are affected by the same signature classes, albeit in different ways.

3 Universal CRT framework for arbitrary composite n

3.1 Signatures and class multiplicities

Let x be a residue of n and for each i with $0 < \alpha_i(x) < a_i$, let $\alpha_i(x)$ be $\min\{v_{p_i}(x), a_i\}$, with $\alpha_i(x) = a_i$ when $\alpha_i(x) = 0$. Define $\alpha_i(x) = \min\{v_{p_i}(x), a_i\}$ for a residue x of n , and $\alpha_i(x) = a_i$ when $\alpha_i(x) = 0$. Thus $\alpha(x)$ lies in $\Pi_i\{0, \dots, a_i\}$. The all-zero vector is a unit and the vector a is zero, hence neither occurs in the list of nonzero zero-divisors. For each other signature α , let $C_{-\alpha} = \{x \in \mathbb{Z}^*(\mathbb{Z}_n) : \alpha(x) = \alpha\}$. Let $C_{-\alpha} = \{x \in \mathbb{Z}^*(\mathbb{Z}_n) : \alpha(x) = \alpha\}$ whenever α is a signature different from the one mentioned above.

Proposition 3.1. The class size is $s_{-\alpha} = \prod_i c_i(\alpha_i)$, where $c_i(j) = p_i^{a_i - j - 1} (p_i - 1)$ for $0 \leq j < a_i$ and $c_i(a_i) = 1$. The number of nonempty signature classes is $\prod_i (a_i + 1) - 2 = \tau(n) - 2$.

Proof. In $\mathbb{Z}_{-(p_i^{a_i})}$, only the valuations of zero have been truncated, and exactly $p_i^{a_i - j} - p_i^{a_i - j - 1}$ residues have valuation $j < a_i$. With CRT, the choices of the coordinates are independent and thus their counts multiply. The number of classes will be the same as above, unless the unit and zeros signatures are excluded.

The $\gcd(x, n)$ in $d = \gcd(x, n)$ is referred to as a scalar divisor and it is the same as the signature, but the vector form shows which prime-power coordinate is the cause of or the obstruction for adjacency. Also, it makes the factor-wise sensitivity obvious: adding one to the exponent a_i adds one level in that coordinate, and does not change the definitions of the other coordinates.

3.2 Compatibility and generalized join

Theorem 3.2 (CRT compatibility). If $x \in C_{-\alpha}$ and $y \in C_{-\beta}$, then $xy \equiv 0 \pmod{n}$ if and only if $\alpha_i + \beta_i \geq a_i$ for every i .

Proof. xy is congruent to 0 mod n if and only if $p_i^{a_i}$ is a factor of xy for every i . The sum of the two valuations will not change by truncation, whether or not they will reach a_i . Therefore, the product is only missing where the coordinate wise inequalities are missing.

Let Q_n be a grid with vertices corresponding to the distinct non-trivial signatures and connect different α, β for $\alpha + \beta \geq a$ coordinate-wise. Let $\epsilon_{-\alpha} = 1$ if $2\alpha \geq a$ coordinate wise, and $\epsilon_{-\alpha} = 0$ otherwise. Define $\epsilon_{-\alpha}$ to be 1 if $2\alpha \geq a$ coordinate wise, and 0 otherwise. By Theorem 3.2 we may say that every pair of classes is either totally joined or totally nonadjacent, and further, that $G_n[C_{-\alpha}]$ is $K_{(s_{-\alpha})}$ if $\epsilon_{-\alpha} = 1$ and is empty graph of order $s_{-\alpha}$ otherwise.

Corollary 3.3 (lossless reconstruction). G_n is the Q_n -generalized join of the graphs $H_{-\alpha}$, where $H_{-\alpha} = K_{(s_{-\alpha})}$ for $\epsilon_{-\alpha} = 1$ and $H_{-\alpha}$ is edgeless for $\epsilon_{-\alpha} = 0$.

Proof. Compile a list of all quotient vertexes α , replacing each one by $H_{-\alpha}$. Condition $2\alpha \geq a$ is the source for the internal edges, and condition $\alpha + \beta \geq a$ is the source for the cross edges. These are precisely all the edges of G_n . $\square$

$$|E(G_n)| = \sum_{(\alpha < \beta, \alpha + \beta \geq a)} s_{-\alpha} s_{-\beta} + \sum_{(\alpha: 2\alpha \geq a)} s_{-\alpha} (s_{-\alpha} - 1) / 2. \quad (1)$$



Figure 1: The same factorization-driven pipeline applies to every composite modulus, regardless of the number or multiplicity of its prime factors.

4 Spectral transfer for general composite moduli

Block matrices and generalized joins have been applied to specific factorization families in some of the spectral work performed on $\Gamma(\mathbb{Z}_n)$ [8, 12, 17, 19, 23, 24, 28]. In this case, it is to be denoted the transfer for arbitrary n in the same notation as distance and optimization will be later. Define the signature classes of the signature function $q=\tau(n)-2$ and set aside symmetric matrices B_A and B_L with entries equal to the values of $q \times q$ matrices A and L by

$$(B_A)_{(\alpha\alpha)} = \epsilon_{\alpha}(s_{\alpha}-1), \quad (B_A)_{(\alpha\beta)} = \sqrt{s_{\alpha}s_{\beta}} \text{ if}$$

$$\alpha \sim \beta, \text{ and } 0 \text{ otherwise,} \quad (2)$$

$$(B_L)_{(\alpha\alpha)} = \Sigma_{\beta}(\beta \sim \alpha) s_{\beta}, \quad (B_L)_{(\alpha\beta)} = -\sqrt{s_{\alpha}s_{\beta}}$$

$$\text{if } \alpha \sim \beta, \text{ and } 0 \text{ otherwise.} \quad (3)$$

Theorem 4.1 (adjacency spectrum). Adjacency spectrum of G_n is the spectrum of the operator B_A together with 0 of multiplicity $\Sigma_{\alpha}(\epsilon_{\alpha}=0)(s_{\alpha}-1)$ and -1 of multiplicity $\Sigma_{\alpha}(\epsilon_{\alpha}=1)(s_{\alpha}-1)$.

Proof. Make use of normalized class-indicator vectors $u_{\alpha} = 1_{(C_{\alpha})} / \sqrt{s_{\alpha}}$. They are fixed under $A(G_n)$ and their representing matrix is B_A . It is orthogonal and its orthogonal complement decomposes into zero-sum subspaces supported on the separate classes. This subspace is such that an independent block is zero and a clique block $J-I$ is $-I$. The dimensions are $s_{\alpha}-1$.

Theorem 4.2 (Laplacian spectrum). The spectrum of B_L together is the Laplacian spectrum, for each class α , that is the spectrum of the graph consisting of $\lambda_{\alpha} = \Sigma_{\beta}(\beta \sim \alpha) s_{\beta} + \epsilon_{\alpha} s_{\alpha}$ repeated $s_{\alpha}-1$ times.

Proof. $D-A$ is the class which yields a reduced matrix B_L when the normalized class-indicator space is used. The cross-class degree adds a sum of the $\beta \sim \alpha$ degrees of the s_{β} 's. The Laplacian of a clique C_{α} contributes s_{α} if C_{α} is a clique otherwise it contributes nothing. This verifies the given eigenvalues and multiplicities.

The formulas are for two algebraic effects. Class multiplicity gives rise to the repeated eigenvalues 0, -1 or λ_{α} , with the interactions between different prime-power signatures being concentrated in q -dimensional matrices. Therefore, without diagonalizing the $n-\varphi(n)-1$ dimensional matrix, the spectral radius, energy and algebraic connectivity can be computed. The outcome is compatible with generalized-join theory [10, 15] whilst retaining the arithmetic meaning of each block.

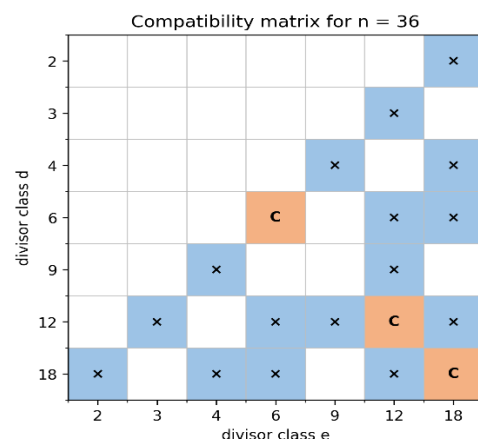


Figure 2: Compatibility of the seven divisor classes for $n=36$. A cross denotes a complete join; C marks an internally complete class.

5 Distances and resolving structure

5.1 Distance and Wiener index

Zero-divisor graphs have a diameter of 3 or greater [6, 18]. The quotient is used to determine exactly where these temporary paths are found. Let d_Q be the shortest-path distance in Q_n . Any class path lifts to a residue path, as all of the indicated cross joins are complete; projecting a residue path results in a quotient walk, for distinct classes α and β .

Theorem 5.1 (distance transfer). For $x \in C_{\alpha}$ and $y \in C_{\beta}$ with $\alpha \neq \beta$, $d_G(x,y) = d_Q(\alpha,\beta)$. For distinct $x,y \in C_{\alpha}$, their distance is one if $\epsilon_{\alpha}=1$ and two if $\epsilon_{\alpha}=0$.

Proof. The first statement is followed by the lifting and projecting of paths. Clique type provides the adjacency in one class. Consider independent type, x , y , and G_n is connected; therefore there exists a common neighbour

giving a path of length two and no edge between x and y .

$$W(G_n) = \sum_{\alpha} \binom{s_{\alpha}}{2} (2 - \epsilon_{\alpha}) + \sum_{(\alpha < \beta)} s_{\alpha} s_{\beta} d_Q(\alpha, \beta). \quad (4)$$

One of the properties that is made explicit by equation (4) is that interaction between the CRT depends on multiplicity and type but cross-class interaction is a weighted distance sum on Q_n . It also yields the entire distance matrix in block form, making distance-Laplacian studies, like [25] possible without having to list all the residue products first.

5.2 Metric dimension and twin classes

The vertices of each independent signature class are false twins, and the vertices of each clique class are true twins (equal neighborhoods (N) outside of the class). All but one vertex in each twin class is to be included in any resolving set. This is common when finding issues and particularly so here as the class sizes are clear [11, 26, 27].

Proposition 5.2 (CRT twin bound). $\dim(G_n) \geq \sum_{\alpha} \max\{s_{\alpha}-1, 0\} = |V(G_n)| - (\tau(n)-2)$.

Proof. If two different vertices u and v of the same class were both not in a resolving set S , then all the landmarks not in u or v would be equally distant from both u and v . The class will assume that there is no landmark in the class. So u and v would be metrically the same which is impossible. The bound is obtained by adding up the class wise requirements.

The bound does not have to be indicated as a universal equality with an additional quotient-separation argument. In Section 8, exact values are obtained by the standard set-cover formulation: add $z_w \in \{0, 1\}$ and enforce that for each unordered pair $\{u, v\}$ of points with $d(u, w) \neq d(v, w)$, $\sum_w z_w \geq 1$; minimize $\sum_w z_w$. Distances are given using Theorem 5.1. This is different from the test based equality to the benchmark and removes the need for extrapolation beyond the tested moduli.

6 Domination and coloring on the same quotient

6.1 Exact class-level domination

Twins and domination don't play the same game. A clique class dominates by the vertex selected by it but an independent class does not dominate by the vertex selected by it. Let t_{α} be the number of vertices selected and z_{α} be 1 if $t_{\alpha} > 0$ and z_{α} be 0 otherwise for each α . This compact model has just that same attribute.

$$\text{Minimize } \sum_{\alpha} t_{\alpha} \quad (5)$$

$$\begin{aligned} z_{\alpha} &\leq t_{\alpha} \leq s_{\alpha} z_{\alpha}, t_{\alpha} + \\ s_{\alpha}(\epsilon_{\alpha} z_{\alpha} + \sum_{(\beta \sim \alpha)} z_{\beta}) &\geq s_{\alpha}, \end{aligned} \quad (6)$$

Integer $0 \leq t_{\alpha} \leq s_{\alpha}$ and binary z_{α} . The second inequality states that whenever a class has no neighbour class activated it should either activate itself as a clique or choose all vertices when it is independent.

Theorem 6.1. The optimum of (5)-(6) equals $\gamma(G_n)$.

Proof. For a dominating set, count the number of vertices selected in each class, get t and z . If a vertex is not selected, then it must appear in the clique class of an activated adjacent class, or in the clique class of a selected vertex, thus (6) is satisfied. On the other hand a feasible solution chooses t_{α} arbitrary vertices of C_{α} . If $t_{\alpha} < s_{\alpha}$, then the selected neighbor is guaranteed to be in a clique class or a fully joined class. So each vertex is a dominated vertex and goals coincide.

6.2 Demand-preserving coloring

A coloring of Q_n is simple if it does not indicate that a clique class needs s_{α} different colours. If $\epsilon_{\alpha} = 1$ then attach $w_{\alpha} = s_{\alpha}$, otherwise $w_{\alpha} = 1$. A weighted multicoloring assigns a set S_{α} of w_{α} colors to every quotient vertex, such that the sets of colors assigned to adjacent quotient vertices are disjoint. Weighted coloring techniques and integer programming are well known [20, 21].

Theorem 6.2. $\chi(G_n)$ equals the minimum number of colors in a feasible multicoloring of (Q_n, w) .

Proof. Only every class get a proper coloring of G_n . A clique class gets s_α colors; an independent class can get recolored with one of its own colors; complete joins require disjoint class palettes. On the other hand color the vertices of a clique with different colors from S_α and color each vertex of an independent class with its own color. Internal and cross-class edges are then correctly coloured.

The domination model and the coloring model share the same Q_n but give different meaning to a class. Domination requires an Activation Indicator, and a Selection Count; Coloring requires a Demand. This is the real-life use of integration: quotient adjacency is calculated only once, whereas the information for each invariant is attached easily and efficiently without the need for separate constructions at the residue level. Here, coloring bounds can be further tightened by means of perfectness results for selected modular families [30] but they are not assumed here.

7 General-composite algorithm and verification

The implementation does not require a named factorization family, but will accept any composite n . It factors n and directly forms an r -dimensional signature space, removes only the signatures $(1, 0)$ and $(0, 1)$, and forms s_α , ε_α and Q_n . The same code path is thus utilised for prime powers, 2-prime products and moduli of 3 or more different prime factors. The reduced spectral matrices are generated by equations (2)-(3); distances sums by (4); domination by (5)-(6); and χ by weighted quotient coloring. Full residue

enumeration is only used as an independent path of validation.

Algorithm 1: Universal CRT construction for composite n

- Factor $n = \prod p_i^{a_i}$ and list other signatures $\alpha \neq 0, \alpha \neq a$.
- Compute each class size as per Proposition 3.1 and type of class as per $2\alpha \geq a$.
- Construct Q_n coordinatewise, that is, ensure that $\alpha + \beta \geq a$. (1)
- Diagonalize B_A and B_L and add the repeated block eigenvalues.
- Calculate the quotient distances $W(G_n)$, the twin bound and the resolving-set model.
- Solve class domination model and demand preserving coloring model.
- Compare edge sets, spectra and reported values of invariants with the residue graph.

The exhaustive domain contains all 138 composite integers from 4 through 180: 14 prime powers, 98 moduli with two distinct prime factors and 26 with at least three. All the adjacency relations and both the spectral decompositions matched with complete residue matrices with an error in the maximum eigenvalue of less than 10^{-7} . The average reduction in the quotient in this domain was 85.6%. Because there is only one residue vertex and one class in the exceptional case $n=4$, there is no reduction; for $n \geq 6$, the reduction is between 33.3% and 97.8%. The more expensive exact invariants are retained as readable representatives, with ten moduli.

Table 2: Reproducibility and verification design

Stage	Reduced size	Independent verification
Adjacency	q classes; $O(q^2)$ tests	all residue pairs
Spectra	$q \times q$ matrices + repeats	full eigendecomposition
Distances	q -class shortest paths	breadth-first search from each residue
Domination	$2q$ class variables	full closed-neighborhood cover
Coloring	q vertices with demands	exact full-graph DSATUR
Metric dimension	quotient-generated distances	full pair-separation cover

8 Outcomes for general composite integers

8.1 Representative structural and spectral outcomes

Table 3: Structural and spectral results

n	factorization	V	q	red.%	E	Q	μ_2
12	$2^2 \cdot 3$	7	4	42.9	8	2.613	0.677
18	$2 \cdot 3^2$	11	4	63.6	13	3.400	0.557
20	$2^2 \cdot 5$	11	4	63.6	16	3.696	0.803
24	$2^3 \cdot 3$	15	6	60.0	26	4.851	0.711
27	3^3	8	2	75.0	13	4.000	2.000
30	$2 \cdot 3 \cdot 5$	21	6	71.4	38	5.637	0.647
32	2^5	15	4	73.3	23	5.029	1.000
36	$2^2 \cdot 3^2$	23	7	69.6	46	6.128	0.761
42	$2 \cdot 3 \cdot 7$	29	6	79.3	56	6.758	0.621
60	$2^2 \cdot 3 \cdot 5$	43	10	76.7	120	9.521	0.803

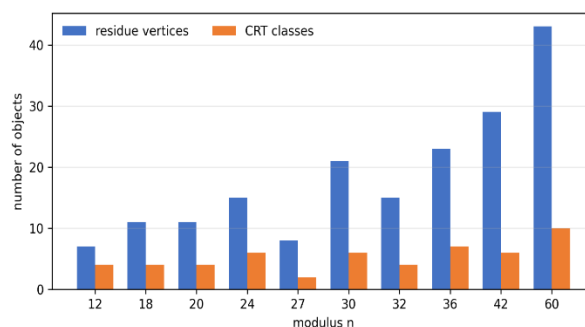


Figure 3: Residue-level order and CRT quotient order for the ten benchmark moduli.

Table 3 is a representative sample of the general-composite domain that is a full cross-section. The reduction values range from 42.9% to 79.3% whereas the full set of 138 moduli has a mean value of 85.6% and a maximum value of 97.8%. It is not n , but the quantities $n-\varphi(n)-1$ and $\tau(n)-2$ that are decisive. Nested valuation layers are produced by prime powers, adding different prime numbers adds a dimension to the signature space, and repeated exponents add dimensions to the individual coordinates. All three changes are allowed and don't change the construction rules of the algorithm.

The tested set shows that the spectral radius grows from 2.613 ($n=12$) to 9.521 ($n=60$), indicating a general tendency for compatibility growth, not just quotient order. Algebraic connectivity is positive and hence the graph is connected, but is small for some mixed factor cases. For example, $\mu_2=0.621$ for $n=42$, whereas

$\mu_2=2$ for $n=27$. The distinction can be traced back to the quotient: $n=27$ shows nested valuation layers with a rather well connected upper-class while $n=42$ reveals three CRT coordinates with several large independent classes that are connected by less broad compatibility pathways. The interpretations are not obtained from an unexplained numerical eigenvalue list, but from B_L .

8.2 Distance, domination, coloring and resolution

Table 4: Metric, domination, coloring and energy results

n	diam.	W	γ	χ	twin LB	dim	energy
12	3	38	2	2	3	3	7.391
18	3	109	2	3	7	7	10.449
20	3	102	2	2	7	7	10.453
24	3	208	2	3	9	9	15.663
27	2	43	1	3	6	6	8.000

n	diam.	W	γ	χ	twin LB	dim	energy
30	3	438	3	3	15	15	19.226
32	2	187	1	4	11	11	13.371
36	3	520	2	5	16	16	23.696
42	3	864	3	3	23	23	23.424
60	3	1958	3	3	33	33	41.889

The diameters of all the mixed factor benchmark numbers is three while the tested prime powers $n=27$ and 32 have diameter two. Diameter has nothing to do with the weighted Wiener index; $n=24$ and $n=32$ have the same order, but $W=208$ and 187 , respectively, due to the difference in quotient-distance distributions. The difference may be due to the difference between pairs of signatures for the same class of objects or it may be due to the difference between paths among different signatures. Equation (4) indicates this.

Even if the metric dimension is big, but the domination is small. In each of these cases, $\Gamma(\mathbb{Z}_{27})$ and $\Gamma(\mathbb{Z}_{32})$ are dominated by a single high-valuation class with $\gamma = 1$. In the cases tested

Table 5: General-composite coverage and principal outcomes

Factorization domain	Cases	Reduction outcome
All composite $4 \leq n \leq 180$	138	mean 85.6%; range 0.0%-97.8%
Prime powers	14	mean 70.2%; range 0.0%-91.7%
Two distinct primes	98	mean 87.1%; range 33.3%-97.8%
At least three primes	26	mean 88.1%; range 71.4%-94.9%
Adjacency and spectra	138	no mismatch; error $< 10^{-7}$
Exact advanced invariants	10 representatives	full and reduced routes agree

9 Generality and algebraic implications

9.1 One construction across factorization families

The signature of the CRT provides an algebraic explanation for each graph-theoretic operation. All of the exponents must be completed to form an edge. If a prime factor is added, thus a new compatibility constraint is added, while if the exponent of a prime factor is increased, then additional valuation layers are added. Class size is multiplicative, that is, changing one prime-powers component reweights the whole cut of the quotient. These weights are stored in the spectral matrices as $\sqrt{s_{\alpha s_{\beta}}}$, used in Wiener

$n=30$ and $n=42$, the three prime factors were selected, and in the case of mixed two-prime powers, two were selected. The case of Metric dimension is opposite as big twin classes are hard to distinguish: $\dim(\Gamma(\mathbb{Z}_{42}))=23$ (but $\gamma=3$). This contrast is an actual ground for not taking an invariant for granted on the basis of another.

In each of the ten cases, the exact metric dimension is equal to the lower bound for twins of the CRT. The outcome is stated as a verified benchmark outcome, rather than a theorem for each modulus. The equality means that, having kept the vertices of each class except one, the vertices that were discarded are spaced apart by the identified landmarks. An overall classification of this is in need of a resolving condition at the level of the quotient and is open. The chromatic numbers are in the range 1539-1549 and are primarily sensitive to the complete joins and clique-class demand as opposed to the class multiplicity of independent blocks.

formula as $s_{\alpha s_{\beta}}$, used in domination as s_{α} , used in coloring only if the class is complete. So, the invariants have the same algebraic structure but are different in responding to it.

The point can be seen in the $n=36$ compatibility matrix. When the unit and zero corners are removed it leaves a two dimensional 3×3 valuation-grid whose factorization is $2^2 \cdot 3^2$. Classes 6, 12 and 18 are internally complete as the two coordinates have passed the half-exponent mark. The upper-right portion of the coordinate wise addition is reserved for cross-class conflicts. If n is changed to 30, both repeated coordinates are replaced by three binary coordinates and all nontrivial classes are thus independent, while the

quotient is changed to have a different 3-compatibility geometry. This is not easily visible when the number of vertices, n , is considered as a parameter only.

9.2 Resolution of the general-composite framework gap

The annihilator compression and the structure of $\Gamma(\mathbb{Z}_n)$ is set up [1, 4, 22, 23, 29, 31]. It is also possible to get spectral, metric and coloring results for important special families [1539-1549, 12, 17, 19, 1539-1549, 30]. The novelty asserted here is at the same time narrower and more general: The paper does not add those invariants, but it eliminates the necessity for a separate structural derivation for each family of factorizations [32,33]. This data object is shown to be sufficient for reconstruction and for all transfer results given in Sections 1539-1549, where s and ϵ are arbitrary vectors with any number of elements.

This addresses the above-mentioned gap at three levels. The coordinates of a CRT signature are algebraically the same as all prime powers. The weighted generalized join is a structural join: it reconstructs the whole residue graph. The algorithms are computed and accept all composite modulus and return the same standardized data. Simple unweighted compressed graph is not enough as it fails to reconstruct repeated eigenvalues, within-class distance pairs, domination costs and clique-class color demand. It is not claimed to be a new ring-isomorphism invariant; rather it is merely a relabeling of a compressed graph.

9.3 Limitations and future research

Firstly, the word "generality" is used for any composite integer n and the ring $\mathbb{Z}_n$, and not for any finite commutative ring. Some other rings might have non-complete, nonempty induced subgraphs for its classes of annihilators. Second, the proofs do not imply a formal complexity advantage in itself; it is rather the correctness of the implementation that is verified exhaustively and the proof is valid for any arbitrary n . Thirdly, metric dimension is bounded generally, and the

equality in Table 4 still needs a characterization on quotient-level.

Other directions are towards normalized, signless, distance and eccentricity spectra, total domination, connected domination, local metric dimension, edge metric dimension, fault-tolerant metric dimension, and list coloring and capacity-constrained coloring. Many of these variants already have specialized zero-divisor results [24, 25, 27] and a helpful next test is whether the matrices or constraints in such variants only depend on (Q_n, s, ϵ) .

10 Conclusion

In the present work, one valuation-signature system is obtained for $\Gamma(\mathbb{Z}_n)$ for each composite number $n = \prod p_i^{a_i}$ without any restriction on the number of distinct primes or the exponents of these primes. Its signature space is determined by the exponent vector, the class sizes are determined by CRT, and the coordinate wise completion of the exponent vector determines the internal and cross class adjacencies. The weighted quotient (Q_n, s, ϵ) then reconstructs the entire graph with accuracy.

The framework gives rise to general adjacency and Laplacian spectral decompositions, a distance-transfer theorem and a Wiener formula, an exact domination model, a bound for the metric dimension and a demand-preserving coloring. No adjacency or spectral mismatch was found for all 138 composite $n \geq 180$ and an average value of structural reduction of 85.6% agreed with exact computations of representative full graphs. The main result is not a formula for a limited family but a procedure which is repeatable and factorization based that handles prime powers, semiprimes, mixed powers of three or more different primes in the same mathematical framework.

11. Acknowledgements

- The authors declare that no external funding was received for this study
- The research work did not receive financial support from any funding agencies.

12. Author Declaration

- This is originality research work

- The authors declare no conflict of interest.

References

- Aijaz, M., Rani, K., & Pirzada, S. (2023). On compressed zero divisor graphs associated to the ring of integers modulo n . *Carpathian Mathematical Publications*, 15(2), 552–558. <https://doi.org/10.15330/cmp.15.2.1539-1549>
- Akbari, S., Maimani, H. R., & Yassemi, S. (2003). When a zero-divisor graph is planar or a complete r -partite graph. *Journal of Algebra*, 270(1), 169–180. [https://doi.org/10.1016/S1539-1549\(03\)01539-1549](https://doi.org/10.1016/S1539-1549(03)01539-1549)
- Anderson, D. F., & Badawi, A. (2008). On the zero-divisor graph of a ring. *Communications in Algebra*, 36(8), 3073–3092. <https://doi.org/10.1080/00927870802110888>
- Anderson, D. F., & LaGrange, J. D. (2012). Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. *Journal of Pure and Applied Algebra*, 216(7), 1626–1636. <https://doi.org/10.1016/j.jpaa.2011.12.008>
- Anderson, D. F., & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *Journal of Algebra*, 217(2), 434–447. <https://doi.org/10.1006/jabr.1998.7840>
- Anderson, D. F., & Mulay, S. B. (2007). On the diameter and girth of a zero-divisor graph. *Journal of Pure and Applied Algebra*, 210(2), 543–550. <https://doi.org/10.1016/j.jpaa.2006.10.007>
- Anderson, D. D., & Naseer, M. (1993). Beck's coloring of a commutative ring. *Journal of Algebra*, 159(2), 500–514. <https://doi.org/10.1006/jabr.1993.1171>
- Bajaj, S., & Panigrahi, P. (2022). On the adjacency spectrum of zero divisor graph of ring $\mathbb{Z}_n$. *Journal of Algebra and Its Applications*, 21, Article 2250197. <https://doi.org/10.1142/S0219498822501973>
- Beck, I. (1988). Coloring of commutative rings. *Journal of Algebra*, 116(1), 208–226. [https://doi.org/10.1016/1539-1549\(88\)91539-1549](https://doi.org/10.1016/1539-1549(88)91539-1549)
- Cardoso, D. M., de Freitas, M. A., Martins, E. A., & Robbiano, M. (2013). Spectra of graphs obtained by a generalization of the join graph operation. *Discrete Mathematics*, 313(5), 733–741. <https://doi.org/10.1016/j.disc.2012.10.016>
- Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. R. (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, 105(1–3), 99–113. [https://doi.org/10.1016/S1539-1549X\(00\)01539-1549](https://doi.org/10.1016/S1539-1549X(00)01539-1549)
- Chattopadhyay, S., Patra, K. L., & Sahoo, B. K. (2020). Laplacian eigenvalues of the zero divisor graph of the ring $\mathbb{Z}_n$. *Linear Algebra and Its Applications*, 584, 267–286. <https://doi.org/10.1016/j.laa.2019.08.015>
- Cockayne, E. J., & Hedetniemi, S. T. (1977). Towards a theory of domination in graphs. *Networks*, 7(3), 247–261. <https://doi.org/10.1002/net.3230070305>
- DeMeyer, F. R., McKenzie, T., & Schneider, K. (2002). The zero-divisor graph of a commutative semigroup. *Semigroup Forum*, 65(2), 206–214. <https://doi.org/10.1007/s002330010128>
- Godsil, C., & Royle, G. (2001). *Algebraic graph theory*. Springer. <https://doi.org/10.1007/1539-1549-1539-1549-9>
- Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. *Discrete Applied Mathematics*, 70(3), 217–229. [https://doi.org/10.1016/1539-1549X\(95\)01539-1549](https://doi.org/10.1016/1539-1549X(95)01539-1549)
- Lande, A., & Khairnar, A. (2024). On the spectrum of generalized zero-divisor graph of the ring $\mathbb{Z}_{p^{\alpha}q^{\beta}}$. *Communications in Mathematics and Applications*, 15(3), 1031–1044. <https://doi.org/10.26713/cma.v15i3.2737>
- Lucas, T. G. (2006). The diameter of a zero divisor graph. *Journal of Algebra*, 301(1), 174–193. <https://doi.org/10.1016/j.jalgebra.2006.01.019>
- Magi, P. M., Jose, M., & Kishore, A. (2020). Adjacency matrix and eigenvalues of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$. *Journal of Mathematical and Computational Science*, 10(4), 1285–1297. <https://doi.org/10.28919/jmcs/4590>

-
- Malaguti, E., Monaci, M., & Toth, P. (2009). Models and heuristic algorithms for a weighted vertex coloring problem. *Journal of Heuristics*, 15(5), 503–526. <https://doi.org/10.1007/s11539-1549-1539-1549>
- Mehrotra, A., & Trick, M. A. (1996). A column generation approach for graph coloring. *INFORMS Journal on Computing*, 8(4), 344–354. <https://doi.org/10.1287/ijoc.8.4.344>
- Pi, S. J., Kim, S. H., & Lim, J. W. (2019). The zero-divisor graph of the ring of integers modulo n . *Kyungpook Mathematical Journal*, 59(4), 591–601. <https://doi.org/10.5666/KMJ.2019.59.4.591>
- Pirzada, S., Aijaz, M., & Bhat, M. I. (2020). On zero-divisor graphs of the rings $\mathbb{Z}_n$. *Afrika Matematika*, 31, 727–737. <https://doi.org/10.1007/s11539-1549-01539-1549>
- Pirzada, S., Rather, B. A., Chishti, T. A., & Samee, U. (2021). On normalized Laplacian spectrum of zero divisor graphs of commutative ring $\mathbb{Z}_n$. *Electronic Journal of Graph Theory and Applications*, 9(2), 331–345. <https://doi.org/10.5614/ejgta.2021.9.2.7>
- Pirzada, S., Rather, B. A., & Chishti, T. A. (2021). On distance Laplacian spectrum of zero divisor graphs of $\mathbb{Z}_n$. *Carpathian Mathematical Publications*, 13(1), 48–57. <https://doi.org/10.15330/cmp.13.1.1539-1549>
- Pirzada, S., & Raja, R. (2017). On the metric dimension of a zero-divisor graph. *Communications in Algebra*, 45(4), 1399–1408. <https://doi.org/10.1080/00927872.2016.1175602>
- Raja, R., Pirzada, S., & Redmond, S. P. (2016). On locating numbers and codes of zero-divisor graphs associated with commutative rings. *Journal of Algebra and Its Applications*, 15(1), Article 1650014. <https://doi.org/10.1142/S0219498816500146>
- Rather, B. A., Pirzada, S., Naikoo, T. A., & Shang, Y. (2021). On Laplacian eigenvalues of the zero-divisor graph associated to the ring of integers modulo n . *Mathematics*, 9(5), Article 482. <https://doi.org/10.3390/math9050482>
- Singh, P., & Bhat, V. K. (2020). Zero-divisor graphs of finite commutative rings: A survey. *Surveys in Mathematics and Its Applications*, 15, 371–397. https://www.utgjiu.ro/math/sma/v15/a15_14.html
- Smith, B. (2016). Perfect zero-divisor graphs of $\mathbb{Z}_n$. *Rose-Hulman Undergraduate Mathematics Journal*, 17(2), Article 6. <https://scholar.rose-hulman.edu/rhumj/vol17/iss2/6>
- Spiroff, S., & Wickham, C. (2011). A zero-divisor graph determined by equivalence classes of zero-divisors. *Communications in Algebra*, 39(7), 2338–2348. <https://doi.org/10.1080/00927872.2010.488675>
- Kannan, M., Sasikala, A. (2026). Major Invariants of Zero-Divisor Graphs of $\mathbb{Z}_n$: A Unified PrimeFactorization Framework. *ZHUZAO/FOUNDRY*, 29(9), 1539-1549. DOI:10.29014/FJ-1539-1549-4977.89.2446.
- Kannan, M., Sasikala, A. (2026). Zero Divisor Graphs $\Gamma(\mathbb{Z}_n)$ for Complex Systems Modelling: Structural Characterization, Graph Invariants and a CRT-Based Framework. *MSW Management*, 36(1), 1539-1549. <https://doi.org/10.7492/80kxvf45>.
-