An LR Method Approach to Transient Analysis of M/M/1 Jackson Type Queueing Network Systems

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ABSTRACT:

The effective analysis of queueing networks under stochastic situations is very important for understanding congestion, waiting time and service performance of real-life systems. In this study, an approach for transient analysis of M/M/1 Jackson type network queueing system with fuzzy parameters is proposed based on LR method. The suggested model is composed of two service nodes and one Central Server with an arrival stream following the Poisson process and service time follows the exponential distribution.

To represent uncertainty in arrival and service rates, triangular and trapezoidal fuzzy numbers are considered. The lower and upper bounds of the performance measures based on α-cut are transformed in Left–Right (LR) form by central values, left spreads and right spreads. The transient state probabilities are derived by solving Kolmogorov forward differential equations, numerically by Runge–Kutta method. The expected system length and mean waiting time key performance measures are calculated for Node-1, Node-2 and Central Server. From the results, it is noted that the LR method is one of the simple, compact and meaningful representation of the fuzzy queueing outcomes. The left and right spreads decrease progressively with the increase of the α-cut value, which means that the uncertainty is reduced and the values are moved towards crisp values. The Central Server exhibits more variability because it has both direct and routed arrivals.

Thus, the LR method is considered to be an effective method to simplify fuzzy calculations and to understand the transient queueing system performance under fuzzy environment.

Keywords: LR method, Jackson Type Queuing network, M/M/1 queue, Transient analysis, Fuzzy queueing system, α-cut.

1. Introduction

Queueing theory is a very common method used for the analysis of congestion, waiting time, length of queues, and efficiency of service in systems where the arrivals and the services are random. The classical M/M/1 queueing model is based on the assumption of: 1. Poisson arrivals 2. Exponential service times and 3. Single service facility. But many real-world systems are composed of service stations, in which customers, jobs, packets or tasks pass from one service station to another. An important foundation for such queueing networks was introduced by Jackson who demonstrated that queueing networks with exponential service stations can be analysed by product-form probability structures [1]. In the years that followed, queueing network models for open, closed, and mixed systems were further extended and applied in computer systems, communication networks, production systems, healthcare systems, and organizational service networks [2–5]. Uncertainty of arrival rates and service rates are common in real world queueing system and may impact the measures of performance, such as expected queue length, mean waiting time, congestion level, and server utilisation. As these parameters are not always accurately known, use of fuzzy set theory is useful for representing imprecise and vague information [6]. The fuzzy concepts were extended to decision making under uncertainty by Bellman and Zadeh [7] and important fuzzy-number operations and fuzzy arithmetic methods were developed by Dubois and Prade [8,9]. Later, fuzzy queueing models were developed to analyse the queueing systems with uncertain parameters, and the most popular method for transforming fuzzy arrival and service rate is to obtain the lower and upper interval bounds by using α-cut representation [10-16].

Transient analysis is crucial since queueing systems may not be in steady state. In many practical scenarios, the length of the queue and the waiting time vary with time until reaching equilibrium. The transient behaviour of the M/M/1 queue and Jackson networks has been studied by several researchers, and it has been demonstrated that transient analysis of queues is essential in understanding the behaviour of queues over short time periods and the performance of the system [17–19]. Hence, transient analysis of an M/M/1 Jackson network of queues is important to investigate the time varying expected queue-lengths and waiting times at each node. Fuzzy and intuitionistic fuzzy models can be used to present uncertainty using α-cut and β-cut approaches, but the interval valued performance measures can sometimes be hard to compare directly. To further handle uncertainty modelling, Atanassov's intuitionistic fuzzy set theory takes into account not only membership and non-membership degrees, but also hesitation degree [20]. Such fuzzy results can be simplified by using the Left–Right (LR) method for representation of a fuzzy number, which is a fuzzy number represented by a central value and a left and right spread. In this way, the LR method can be used as a compact and structured method for analysing the fuzzy performance measures derived from a lower bound and an upper bound. Originality of the proposed work is the combination of four important things in queueing, namely transient queueing analysis, Jackson network structure, fuzzy parameter modelling and LR representation. Most of the related queueing studies have been conducted in one of three ways: either focusing on crisp Jackson networks, fuzzy queueing models, or uncertainty analysis based on intervals. However, the present study combines these aspects into a unified framework for analysing an M/M/1 Jackson-type network queueing system under uncertain arrival and service rates. By applying the LR method to α-cut based fuzzy performance measures, the study converts interval-valued outputs into centre values with left and right spreads, making the results more compact and interpretable. Previously published fuzzy queueing models have mainly used α-cut or parametric programming approaches to convert fuzzy queues into equivalent crisp queueing problems and then derive fuzzy performance measures such as queue length and waiting time in interval or membership-function form [11,13,14,15]. Similarly, transient Jackson network studies have focused on time-dependent behaviour and approximation of Jackson networks, but generally under crisp stochastic parameters [19]. In contrast, the present study combines transient Jackson-type network analysis, fuzzy arrival and service parameters, Runge–Kutta numerical solution, and LR representation in a single framework. Therefore, the proposed method differs from earlier models by transforming α-cut based interval outputs into centre value, left spread, and right spread, which makes the transient fuzzy queueing results more compact, interpretable, and easier to compare across Node-1, Node-2, and the Central Server. The specific contribution of this study is that it extends LR representation techniques to the transient analysis of fuzzy M/M/1 Jackson network queueing systems. This advances the queueing theory in the direction of better analysis of interconnected service systems under uncertainty and to the areas of fuzzy modelling as it simplifies the interval results based on α-cut and of applied mathematics as it provides a compact centre-spread formulation, and to the area of numerical analysis as it links the Runge–Kutta based transient solution with LR based interpretation of the fuzzy performance measures. The study focuses on 1). To present the concept of LR (Left–Right) representation of fuzzy numbers. 2. To convert the lower/upper bounds based on an α–cut into LR form using center and spread. 3. To calculate the centre (C), left spread (WL) and right spread (WR) of the system performance measures, like Expected queue length (LN) at node-1, node-2 and Central server, Mean waiting time (WN) at node-1, node-2 and Central server. This method allows understanding the behaviour of fuzzy transient queueing systems, which involve uncertain arrival and service parameters, in terms of variability, asymmetry and convergence behaviour.

2. Materials and Methods

2.1. Research Design

This study investigates the transient behaviour of an M/M/1 Jackson type network queueing system consisting of two service nodes and one Central Sever. To account for uncertainty in arrival and service rates, triangular and trapezoidal fuzzy numbers are incorporated into the model. The LR method is employed to transform fuzzy performance measures into a compact and interpretable form.

2.2. Queueing Network Model

We considered the Transient Analysis of Jackson type Network Queue consists of two nodes and one Central Server with Task Abandonment as detailed below:

1. The capacity of each node as well as Central Server are assumed as S(finite)
2. The Task Scheduling is allowed from nodes to the Central Server whereas flow from one node to another node is restricted. Also assumed that customers can directly enter in to Central server queue. The mean arrival rates at node-1, node-2 and Central server are assumed to follow Poisson Process with respective mean arrival rates $\lambda_1, \lambda_2\ and\ \lambda_3$.
3. The mean service rates for first essential services at node-1, node-2 and Central Server are $\mu_1, \mu_2\ and\ \mu_3$.
4. Customers at nodes who want additional service from Central Server can opt with optional probabilities of $p_1\ and\ p_2$ respectively.
5. Task Abandonment is characterized by two sets of parameters: the balking parameters ($1-b_1$), ($1-b_2$) and ($1-b_3$) representing the probabilities that a customer declines to join the queues of node-1, node-2 and Central server; and the reneging parameters $\xi_1, \xi_2\ and\ \xi_3$, representing the rate at which customers abandon the respective queues after waiting.
6. The system of Kolmogorov forward ordinary differential equations is solved numerically using the fourth-order Runge-Kutta (RK4) method in MATLAB. The computed transient probabilities are subsequently used to evaluate the expected queue lengths and mean waiting times.
7. The transient state probabilities were updated by using Runge-Kutta method of order 4 with time step size of 0.05($\Delta t = 0.5$), truncation error of $O(\Delta t^4)$ and convergence tolerance of 10^{-6}.
8. The computation was stopped at the final time point or when the difference between

successive performance measures became negligible. For validation, the fuzzy parameters were reduced to their crisp/modal values and the resulting behaviour was compared with the standard M/M/1/Jackson network pattern.

With the above assumptions and parameter definitions, an infinitesimal generator matrix Q is obtained from derived Kolmogorov forward equations (In order to describe how the state probabilities evolve over time given in appendix.) as follows:

2.3 Infinitesimal Generator Matrix Q Formulation

Let

$$X(t) = \{N_1(t), N_2(t), N_3(t)\}$$

denote the state of the system at time *t*, where $N_1(t)$, $N_2(t)$, and $N_3(t)$ represent the number of tasks at Node-1, Node-2, and the Central Server, respectively. Since the capacity of each service

station is finite,

$$0 \le N_1, N_2, N_3 \le S.$$

The state space of the system is defined as

$$\Omega = \{(i, j, k) : 0 \le i, j, k \le S\}$$

with cardinality

$$\mid \Omega \mid = (S + 1)^3.$$

Define

$$P(t) = [P_{i,j,k}(t)]_{(i,j,k)\in\Omega}.$$

denote the transient probability vector of the system. Then, the transient behaviour of the

system is governed by the Kolmogorov forward equation

$$\frac{dP(t)}{dt} = P(t)Q$$

where

$$Q = \left[q_{(i,j,k),(r,s,l)}\right]$$

is an $(S+1)^3 \times (S+1)^3$ infinitesimal generator matrix.

2.3.1 Structure of Q

The generator matrix Q has a block tridiagonal structure with respect to the Central Server population level *k*, and can be written as

$$Q = \begin{bmatrix} D_0 & A_0 & 0 & 0 & \cdots & 0 \\ B_1 & D_1 & A_1 & 0 & \cdots & 0 \\ 0 & B_2 & D_2 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_S & D_S & \end{bmatrix}$$

where each block is of dimension

$$(S+1)^2 \times (S+1)^2.$$

Here

- A_k: transitions increasing the Central Server population $(k \to k+1)$
- B_k: transitions decreasing the Central Server population $(k \to k-1)$
- D_k: transitions within the same Central Server level k

2.3.2 Generic Matrix Elements

For any state $(i, j, k) \in \Omega$, the non-zero off-diagonal elements of Q are defined as follows.

External Arrivals

$$q_{(i,j,k),(i+1,j,k)} = \lambda_1 b_1, \qquad i < S$$
$$q_{(i,j,k),(i,j+1,k)} = \lambda_2 b_2, \qquad j < S$$
$$q_{(i,j,k),(i,j,k+1)} = \lambda_3 b_3, \qquad k < S$$

Service Completions and Reneging at Node-1

For i ≥1, a task may leave Node-1 after service completion without routing to the Central server. In addition, if i ≥ 2, waiting tasks may abandon the queue due to reneging.

Therefore,

$$q_{(i,j,k),(i-1,j,k)} = (1-p_1)\mu_1 + (i-1)\,\xi_1, \quad i \ge 1$$

When $i = 1$, the reneging term becomes zero.

Service Completions and Reneging at Node-2

Similarly, for Node-2,

$$q_{(i,j,k),(i,j-1,k)} = (1-p_2)\mu_2 + (j-1)\,\xi_2, \quad j \ge 1$$

When $j = 1$, the reneging term becomes zero.

Service Completions and Reneging at the Central Server

For the Central Server,

$$q_{(i,j,k),(i,j,k-1)} = \mu_3 + (k-1)\,\xi_3, \qquad k \geq 1$$

When $k = 1$, the reneging term becomes zero.

Routing from Nodes to the Central Server

After service completion at Node-1, a task may be routed to the Central Server with probability p_1. Therefore,

$$q_{(i,j,k),(i-1,j,k+1)} = p_1\mu_1, \qquad i \geq 1, k < S.$$

Similarly, after service completion at Node-2, a task may be routed to the Central Server with probability p_2. Therefore,

$$q_{(i,j,k),(i,j-1,k+1)} = p_2\mu_2, \qquad j \geq 1, k < S.$$

Diagonal Elements

The diagonal elements of the infinitesimal generator matrix are defined as the negative sum of all transition rates out of the state (i, j, k). Thus,

$$q_{(i,j,k),(i,j,k)} = -\sum_{(r,s,l)\neq(i,j,k)} q_{(i,j,k),(r,s,l)}.$$

All other elements of Q are zero. This construction ensures that each row of the generator matrix sums to zero, which is required for a valid continuous-time Markov chain representation of the finite-capacity queueing network.

2.4 Fuzzy Parameter Representation

To incorporate uncertainty in arrival and service rates, triangular and trapezoidal fuzzy numbers are used. The fuzzy arrival rates and service rates are converted into lower and upper bounds using α-cut representation for α values ranging from 0 to 1. In the triangular fuzzy case, each fuzzy parameter is represented by lower, modal, and upper values. In the trapezoidal fuzzy case, each parameter is represented by four values defining the lower support, lower core, upper core, and upper support.

2.5 LR Representation Method

LR (Left–Right) method is an efficient method for representation of fuzzy numbers in fuzzy set theory, which is based on the left spread and right spread of fuzzy numbers about central value. The LR method of fuzzy numbers represents fuzzy numbers by their Center value (C), Left spread (WL) and Right spread (WR).

This representation is particularly convenient in queueing models involving fuzzy parameters like the arrival rate and service rate of customers. It aids to simplify the arithmetic operations of fuzzy quantity, fuzzy ranking and fuzzy comparisons.

L-R Fuzzy Numbers

L-R fuzzy numbers are a generalized representation of fuzzy numbers defined using left (L) and right (R) shape functions. These functions are decreasing and map from [0, ∞) to [0,1]. An L-R fuzzy number is represented as $\bar{M} = \langle m, a, b\rangle$ LR, where: m is the modal (central) value, a is the left spread and b is the right spread.

The functions satisfy:
$L(0) = R(0) = 1$
$L(x) \to 0$ as $x \to \infty$, $R(x) \to 0$ as $x \to \infty$
$L(x), R(x) > 0$ for all $x \geq 0$

1. Operations in L-R Type Triangular Fuzzy Numbers

For $\bar{M} = \langle m, a, b\rangle LR$ and $\bar{N} = \langle n, c, d\rangle LR$:

Addition: $\bar{M} + \bar{N} = \langle m + n, a + c, b + d\rangle LR$

Subtraction: $\bar{M} - \bar{N} = \langle m - n, a + d, b + c\rangle LR$

Multiplication: $\bar{M} \times \bar{N} = \langle mn, mc + na - ac, md + nb + bd\rangle LR$

Division: $\bar{M} / \bar{N} = \langle m/n, (md + ad)/(n(n+d)), (mc + bc)/(n(n-c))\rangle LR$

2. Operations in L-R Type Trapezoidal Fuzzy Numbers

An L-R trapezoidal fuzzy number is defined as $\bar{M} = \langle m, n, \alpha, \beta\rangle LR$. It represents a trapezoidal membership function with left and right spreads.

- For $X = \langle x, p, q, r\rangle$ and $Y = \langle y, s, t, u\rangle$:
- Addition: $X + Y = \langle x + y, p + s, q + t, r + u\rangle$
- Subtraction: $X - Y = \langle x - y, p - s, q + u, r + t\rangle$
- Multiplication: $X \times Y = \langle xy, ps, xt + yq, pu + rs\rangle$
- Division: $X / Y = \langle x/s, p/y, q/u, r/t\rangle$

In this work, the LR method is applied to the α–cut based lower and upper bounds of system performance measures such as Expected queue lengths and Mean waiting times for Node-1, Node-2 and Central Server.

2.6 Performance Measures

Some Queueing constants that are calculated to forecast the system through Runge-Kutta method of order 4(RK4) are listed below:

1. Expected lengths of the Node-1, Node-2 and Central Server ($L_{N1}^{(t)}$, $L_{N2}^{(t)}$ and $L_{HO}^{(t)}$) where,

$L_{N1}^{(t)}$	$L_{N2}^{(t)}$	$L_{HO}^{(t)}$
$L_{N1}^{(t)} = \sum_{i=0}^{S} \sum_{j=0}^{S} \sum_{k=0}^{S} ip_{i,j,k}^{(t)}$	$L_{N2}^{(t)} = \sum_{i=0}^{S} \sum_{j=0}^{S} \sum_{k=0}^{S} jp_{i,j,k}^{(t)}$	$L_{HO}^{(t)} = \sum_{i=0}^{S} \sum_{j=0}^{S} \sum_{k=0}^{S} kp_{i,j,k}^{(t)}$

2. Mean Processing Delays at Node-1, Node-2 and Central server are represented as ($W_{N1}^{(t)}, W_{N2}^{(t)}$ and $W_{HO}^{(t)}$) and are given as

$W_{N1}^{(t)}$	$W_{N2}^{(t)}$	$W_{HO}^{(t)}$)
$W_{N1}^{(t)} = \frac{LN_1(t)}{\lambda_1 b_1}$	$W_{N2}^{(t)} = \frac{LN_2(t)}{\lambda_2 b_2}$	$W_{HO}^{(t)} = \frac{LN_2(t)}{\lambda_3 b_3 + p_1\mu_1 + p_2\mu_2}$

2.7 Computational Details

The fuzzy arrival and service parameters are first represented using α-cut intervals. The lower and upper bounds of the performance measures are then obtained for each α-level. Subsequently, the LR method is applied to determine the centre value, left spread, and right spread, thereby providing a simplified representation of fuzzy results.

- Obtain lower and upper bounds using α–cut (Fuzzy) and/or β–cut (Intuitionistic fuzzy)
- For each *α*: $C = \frac{Lower+Upper}{2}$, $WL = C - Lower$, $WR = Upper - C$

Example: For $\mathrm{LN1_{lower}}$ and $\mathrm{LN1_{upper}}$, compute:

$\mathrm{C_{LN1}} = \frac{\mathrm{LN1_{lower}} + \mathrm{LN1_{upper}}}{2}$, $\mathrm{WL_{LN1}} = \mathrm{C_{LN1}} - \mathrm{LN1_{lower}}$, $\mathrm{WR_{LN1}} = \mathrm{LN1_{upper}} - \mathrm{C_{LN1}}$

Represent fuzzy number as: $A = (C, WL, WR)$

2.8. Validation of the Numerical Model

To validate the proposed fuzzy LR-based transient queueing model, the fuzzy system was reduced to its corresponding crisp M/M/1 Jackson type network form. This was done by considering the modal values of the triangular and trapezoidal fuzzy arrival and service rates. Equivalently, the fuzzy parameters may be converted into crisp parameters by taking the lower, modal, and upper values as equal. Under this condition, the α-cut intervals collapse into single crisp values, and the LR representation gives zero left and right spreads. Thus, the model behaves as a conventional crisp queueing network.

The obtained crisp results were examined with respect to the standard behaviour of M/M/1 and Jackson network queueing systems. The results showed that the expected system length and mean waiting time increased when the arrival rates increased, whereas both performance measures decreased when the service rates increased. This behaviour is consistent with classical queueing theory, where higher traffic intensity produces greater congestion and longer waiting time, while improved service capacity reduces system congestion.

Therefore, the validation confirms that the proposed fuzzy LR-based transient model is mathematically consistent with the classical M/M/1 Jackson type network model when uncertainty is removed. It also supports the reliability of the proposed approach for analysing transient queueing performance under fuzzy arrival and service parameters.

3. Results (Numerical Tables)

- The system of differential equations governing the transient state probabilities is solved numerically using the fourth-order Runge–Kutta method. MATLAB is used to implement the computational algorithm and obtain the required performance measures at different time instances.
- For these numerical illustrations, the values for all the model parameters are chosen that which satisfy traffic intensity and they are

$S=3$, $\lambda_1=0.01$, $\lambda_2=0.015$, $\lambda_3=0.02$, $\mu_1=0.02$, $\mu_2=0.025$, $\mu_3=0.03$, $\xi_1=0.001$, $\xi_2=0.0015$, $\xi_3=0.002$, $b_1=0.5$, $b_2=0.55$, $b_3=0.6$, $p_1=0.001$ and $p_2=0.0015$.

Table 1 : The Pattern of mean lengths and waiting times of each node and Central Server over different time instances with these metrics are resulted in the following table:

Constants	t_1	t_2	t_3	t_4	t_5
$L_{N1}^{(t)}$	0.000000007758	0.000000061633	0.000000206563	0.000000486214	0.000000943000
$L_{N2}^{(t)}$	0.000000034728	0.000000274957	0.000000918369	0.000002154331	0.000004164127
$L_{HO}^{(t)}$	0.000000106481	0.000000839975	0.000002795294	0.000006533215	0.000012581627
$W_{N1}^{(t)}$	0.000024825781	0.000197226692	0.000661002136	0.001555885038	0.003017600169
$W_{N2}^{(t)}$	0.000067352098	0.000533250923	0.001781080336	0.004178099679	0.008075894209
$W_{HO}^{(t)}$	0.000141974189	0.001119966589	0.003727062786	0.008710972066	0.016775572796

We have studied the model in two scenarios [I&II] to explore various patterns of queue constants, where Scenario-I shows the behaviour of queue constants over time in Triangular Fuzzy and Scenarios II deals with the behaviour of queue constants over time in Trapezoidal Fuzzy model using LR method.

3.1. Transient Performance under Triangular Fuzzy Parameters using LR Method

Let us consider an FM/FM/1, Jackson network queueing system with two nodes and one Central Server. Suppose the arrivals rates and service rates are triangular fuzzy numbers and are represented as

$\alpha = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]$

$\lambda_x = [\,0.005\ 0.01\ 0.02]$; $\lambda_y = [\,0.01, 0.015, 0.025]$; $\lambda_h = [\,0.015, 0.02, 0.03]$

$\mu_x = [0.015, 0.02, 0.03]$; $\mu_y = [\,0.02, 0.025, 0.035]$; $\mu_h = [\,0.025, 0.03\ 0.04]$

For a given range of values of $\alpha, \lambda_x, \lambda_y, \lambda_h, \mu_x, \mu_y$ and μ_h; the expected length of the system at each node and Central Server and

waiting times at respective areas are shown from
Tables 3.1.1 to 3.1.3

Table 3.1.1: α-Cut based System Length (LN1) and Waiting Time (WN1) in Triangular Fuzzy using LR Method

α	$LN1_{(Right)}$	$LN1_{(Middle)}$	$LN1_{(Left)}$	$WN1_{(Right)}$	$WN1_{(Middle)}$	$WN1_{(Left)}$
0	3.656×10^{-6}	3.771×10^{-6}	3.656×10^{-6}	5.575×10^{-3}	6.308×10^{-3}	5.575×10^{-3}
0.1	3.108×10^{-6}	3.26×10^{-6}	3.108×10^{-6}	4.919×10^{-3}	5.806×10^{-3}	4.919×10^{-3}
0.2	2.608×10^{-6}	2.806×10^{-6}	2.608×10^{-6}	4.284×10^{-3}	5.341×10^{-3}	4.284×10^{-3}
0.3	2.154×10^{-6}	2.407×10^{-6}	2.154×10^{-6}	3.671×10^{-3}	4.914×10^{-3}	3.671×10^{-3}
0.4	1.743×10^{-6}	2.059×10^{-6}	1.743×10^{-6}	3.081×10^{-3}	4.524×10^{-3}	3.081×10^{-3}
0.5	1.372×10^{-6}	1.761×10^{-6}	1.372×10^{-6}	2.512×10^{-3}	4.172×10^{-3}	2.512×10^{-3}
0.6	1.037×10^{-6}	1.51×10^{-6}	1.037×10^{-6}	1.966×10^{-3}	3.857×10^{-3}	1.966×10^{-3}
0.7	7.36×10^{-7}	1.304×10^{-6}	7.36×10^{-7}	1.442×10^{-3}	3.579×10^{-3}	1.442×10^{-3}
0.8	4.647×10^{-7}	1.14×10^{-6}	4.647×10^{-7}	9.392×10^{-4}	3.339×10^{-3}	9.392×10^{-4}
0.9	2.204×10^{-7}	1.015×10^{-6}	2.204×10^{-7}	4.586×10^{-4}	3.136×10^{-3}	4.586×10^{-4}
1	0	9.284×10^{-7}	0	0	2.971×10^{-3}	0

Table 3.1.2: α-Cut based System Length (LN2) and Waiting Time (WN2) in Triangular Fuzzy using LR Method

α	$LN2_{(Right)}$	$LN2_{(Middle)}$	$LN2_{(Left)}$	$WN2_{(Right)}$	$WN2_{(Middle)}$	$WN2_{(Left)}$
0	8.881×10^{-6}	1.008×10^{-5}	8.881×10^{-6}	9.289×10^{-3}	1.277×10^{-2}	9.289×10^{-3}
0.1	7.694×10^{-6}	9.083×10^{-6}	7.694×10^{-6}	8.244×10^{-3}	1.209×10^{-2}	8.244×10^{-3}
0.2	6.584×10^{-6}	8.183×10^{-6}	6.584×10^{-6}	7.224×10^{-3}	1.145×10^{-2}	7.224×10^{-3}
0.3	5.548×10^{-6}	7.377×10^{-6}	5.548×10^{-6}	6.231×10^{-3}	1.086×10^{-2}	6.231×10^{-3}
0.4	4.58×10^{-6}	6.662×10^{-6}	4.58×10^{-6}	5.263×10^{-3}	1.031×10^{-2}	5.263×10^{-3}
0.5	3.678×10^{-6}	6.034×10^{-6}	3.678×10^{-6}	4.322×10^{-3}	9.806×10^{-3}	4.322×10^{-3}
0.6	2.837×10^{-6}	5.491×10^{-6}	2.837×10^{-6}	3.406×10^{-3}	9.345×10^{-3}	3.406×10^{-3}
0.7	2.053×10^{-6}	5.029×10^{-6}	2.053×10^{-6}	2.516×10^{-3}	8.93×10^{-3}	2.516×10^{-3}
0.8	1.321×10^{-6}	4.645×10^{-6}	1.321×10^{-6}	1.651×10^{-3}	8.558×10^{-3}	1.651×10^{-3}
0.9	6.384×10^{-7}	4.336×10^{-6}	6.384×10^{-7}	8.129×10^{-4}	8.232×10^{-3}	8.129×10^{-4}
1	0	4.099×10^{-6}	0	0	7.95×10^{-3}	0

Table 4.1.3: α-Cut based System Length (LHO) and Waiting Time (WHO) in Triangular Fuzzy using LR Method

α	LHO $_{(Right)}$	LHO$_{(Middle)}$	LHO$_{(Left)}$	WHO $_{(Right)}$	WHO$_{(Middle)}$	WHO$_{(Left)}$
0	1.827×10^{-5}	2.343×10^{-5}	1.827×10^{-5}	1.395×10^{-2}	2.312×10^{-2}	1.395×10^{-2}
0.1	1.599×10^{-5}	2.168×10^{-5}	1.599×10^{-5}	1.242×10^{-2}	2.222×10^{-2}	1.242×10^{-2}
0.2	1.382×10^{-5}	2.009×10^{-5}	1.382×10^{-5}	1.092×10^{-2}	2.138×10^{-2}	1.092×10^{-2}
0.3	1.176×10^{-5}	1.865×10^{-5}	1.176×10^{-5}	9.453×10^{-3}	2.059×10^{-2}	9.453×10^{-3}
0.4	9.81×10^{-6}	1.736×10^{-5}	9.81×10^{-6}	8.014×10^{-3}	1.985×10^{-2}	8.014×10^{-3}
0.5	7.957×10^{-6}	1.62×10^{-5}	7.957×10^{-6}	6.604×10^{-3}	1.916×10^{-2}	6.604×10^{-3}
0.6	6.198×10^{-6}	1.518×10^{-5}	6.198×10^{-6}	5.224×10^{-3}	1.853×10^{-2}	5.224×10^{-3}
0.7	4.528×10^{-6}	1.429×10^{-5}	4.528×10^{-6}	3.874×10^{-3}	1.794×10^{-2}	3.874×10^{-3}
0.8	2.942×10^{-6}	1.353×10^{-5}	2.942×10^{-6}	2.553×10^{-3}	1.741×10^{-2}	2.553×10^{-3}
0.9	1.434×10^{-6}	1.29×10^{-5}	1.434×10^{-6}	1.262×10^{-3}	1.693×10^{-2}	1.262×10^{-3}
1	0	1.238×10^{-5}	0	0	1.651×10^{-2}	0

Interpretation :

- The expected system length at Node-1, 0.000000943000 is lying between 0.000000000000000 to 0.000003656240641 of Right as well as Left spread in Triangular fuzzy using LR Method, similarly expected system wating time at same node, 0.003017600169 is lying between 0.000000000000000 to 0.005575353493652 of Right as well as Left spread.
- The expected system length at Node-2, 0.000004164127 is lying between 0.000000000000000 to 0.0000088812826673 of Right as well as Left spread in Triangular fuzzy using LR Method, similarly expected system wating time at same node, 0.008075894209 is lying between 0.000000000000000 to 0.009288833925485 of Right as well as Left spread.
- The expected system length at Central Server (LHO), 0.000012581627 is lying between 0.000000000000000 to 0.000018269918148 of Right as well as Left spread in Triangular fuzzy using LR Method, similarly expected system wating time at Central Server (LHO), 0.016775572796 is lying between 0.000000000000000 to 0.019948479205407 of Right as well as Left spread.

3.2. Trapezoidal Fuzzy Number

Let us consider an FM/FM/1, Jackson network queueing system with two nodes and one Central Server. Suppose the arrivals rates and service rates are Trapezoidal fuzzy numbers and are represented as

α = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]

λ_x = [0.008, 0.0099, 0.0101, 0.012]; λ_y = [0.013, 0.0144, 0.0152, 0.017];

λ_h = [0.018, 0.0198, 0.0202, 0.022]

μ_x = [0.018, 0.0198, 0.0202, 0.022]; μ_y = [0.023, 0.0248, 0.0252, 0.027];

μ_h = [0.028, 0.0298, 0.0302, 0.032]

For a given range of values of α, λ_x, λ_y, λ_y, μ_x, μ_y and μ_h the expected length of the system at each node and Central Server and waiting times at respective areas are shown in Tables of 3. 2.1 to 3. 2.3

Table 3.2.1: α-Cut based System Length (LN1) and Waiting Time (WN1) in Trapezoidal Fuzzy using LR Method

α	$LN1_{(Right)}$	$LN1_{(Middle)}$	$LN1_{(Left)}$	$WN1_{(Right)}$	$WN1_{(Middle)}$	$WN1_{(Left)}$
0	5.664×10^{-7}	1.041×10^{-6}	5.664×10^{-7}	1.194×10^{-3}	3.092×10^{-3}	1.194×10^{-3}
0.1	5.114×10^{-7}	1.021×10^{-6}	5.114×10^{-7}	1.081×10^{-3}	3.07×10^{-3}	1.081×10^{-3}
0.2	4.567×10^{-7}	1.002×10^{-6}	4.567×10^{-7}	9.673×10^{-4}	3.05×10^{-3}	9.673×10^{-4}
0.3	4.024×10^{-7}	9.859×10^{-7}	4.024×10^{-7}	8.539×10^{-4}	3.033×10^{-3}	8.539×10^{-4}
0.4	3.483×10^{-7}	9.716×10^{-7}	3.483×10^{-7}	7.405×10^{-4}	3.017×10^{-3}	7.405×10^{-4}
0.5	2.945×10^{-7}	9.594×10^{-7}	2.945×10^{-7}	6.271×10^{-4}	3.004×10^{-3}	6.271×10^{-4}
0.6	2.41×10^{-7}	9.492×10^{-7}	2.41×10^{-7}	5.137×10^{-4}	2.993×10^{-3}	5.137×10^{-4}
0.7	1.876×10^{-7}	9.41×10^{-7}	1.876×10^{-7}	4.003×10^{-4}	2.984×10^{-3}	4.003×10^{-4}
0.8	1.343×10^{-7}	9.349×10^{-7}	1.343×10^{-7}	2.87×10^{-4}	2.978×10^{-3}	2.87×10^{-4}
0.9	8.12×10^{-8}	9.308×10^{-7}	8.12×10^{-8}	1.736×10^{-4}	2.973×10^{-3}	1.736×10^{-4}
1	2.811×10^{-8}	9.287×10^{-7}	2.811×10^{-8}	6.024×10^{-5}	2.971×10^{-3}	6.024×10^{-5}

Table 3.2.2: α-Cut based System Length (LN2) and Waiting Time (WN2) in Trapezoidal Fuzzy using LR Method

α	$LN2_{(Right)}$	$LN2_{(Middle)}$	$LN2_{(Left)}$	$WN2_{(Right)}$	$WN2_{(Middle)}$	$WN2_{(Left)}$
0	1.657×10^{-6}	4.321×10^{-6}	1.657×10^{-6}	2.134×10^{-3}	8.095×10^{-3}	2.134×10^{-3}
0.1	1.519×10^{-6}	4.27×10^{-6}	1.519×10^{-6}	1.96×10^{-3}	8.052×10^{-3}	1.96×10^{-3}
0.2	1.382×10^{-6}	4.223×10^{-6}	1.382×10^{-6}	1.787×10^{-3}	8.01×10^{-3}	1.787×10^{-3}
0.3	1.245×10^{-6}	4.178×10^{-6}	1.245×10^{-6}	1.614×10^{-3}	7.971×10^{-3}	1.614×10^{-3}
0.4	1.11×10^{-6}	4.136×10^{-6}	1.11×10^{-6}	1.442×10^{-3}	7.933×10^{-3}	1.442×10^{-3}
0.5	9.763×10^{-7}	4.097×10^{-6}	9.763×10^{-7}	1.27×10^{-3}	7.897×10^{-3}	1.27×10^{-3}
0.6	8.432×10^{-7}	4.061×10^{-6}	8.432×10^{-7}	1.099×10^{-3}	7.864×10^{-3}	1.099×10^{-3}
0.7	7.109×10^{-7}	4.029×10^{-6}	7.109×10^{-7}	9.28×10^{-4}	7.832×10^{-3}	9.28×10^{-4}
0.8	5.796×10^{-7}	3.998×10^{-6}	5.796×10^{-7}	7.576×10^{-4}	7.801×10^{-3}	7.576×10^{-4}
0.9	4.49×10^{-7}	3.971×10^{-6}	4.49×10^{-7}	5.877×10^{-4}	7.773×10^{-3}	5.877×10^{-4}
1	3.193×10^{-7}	3.947×10^{-6}	3.193×10^{-7}	4.182×10^{-4}	7.747×10^{-3}	4.182×10^{-4}

Table 4.2.3: α-Cut based System Length (LHO) and Waiting Time (WHO) in Trapezoidal Fuzzy using LR Method

α	LHO $_{(Right)}$	LHO$_{(Middle)}$	LHO$_{(Left)}$	WHO $_{(Right)}$	WHO$_{(Middle)}$	WHO$_{(Left)}$
0	3.745×10^{-6}	1.276×10^{-5}	3.745×10^{-6}	3.326×10^{-3}	1.668×10^{-2}	3.326×10^{-3}
0.1	3.406×10^{-6}	1.269×10^{-5}	3.406×10^{-6}	3.026×10^{-3}	1.665×10^{-2}	3.026×10^{-3}
0.2	3.067×10^{-6}	1.263×10^{-5}	3.067×10^{-6}	2.727×10^{-3}	1.662×10^{-2}	2.727×10^{-3}
0.3	2.729×10^{-6}	1.258×10^{-5}	2.729×10^{-6}	2.428×10^{-3}	1.66×10^{-2}	2.428×10^{-3}
0.4	2.392×10^{-6}	1.254×10^{-5}	2.392×10^{-6}	2.128×10^{-3}	1.658×10^{-2}	2.128×10^{-3}
0.5	2.055×10^{-6}	1.25×10^{-5}	2.055×10^{-6}	1.829×10^{-3}	1.656×10^{-2}	1.829×10^{-3}
0.6	1.718×10^{-6}	1.246×10^{-5}	1.718×10^{-6}	1.53×10^{-3}	1.654×10^{-2}	1.53×10^{-3}
0.7	1.381×10^{-6}	1.243×10^{-5}	1.381×10^{-6}	1.23×10^{-3}	1.653×10^{-2}	1.23×10^{-3}
0.8	1.045×10^{-6}	1.241×10^{-5}	1.045×10^{-6}	9.31×10^{-4}	1.652×10^{-2}	9.31×10^{-4}
0.9	7.091×10^{-7}	1.239×10^{-5}	7.091×10^{-7}	6.318×10^{-4}	1.651×10^{-2}	6.318×10^{-4}
1	3.732×10^{-7}	1.238×10^{-5}	3.732×10^{-7}	3.325×10^{-4}	1.651×10^{-2}	3.325×10^{-4}

Interpretation :

- The expected system length at Node-1, 0.000000943000 is lying between 0.0000009286769584 to 0.0000010408493839 of Right as well as Left spread in Trapezoidal fuzzy using LR Method, similarly expected system wating time at same node, 0.003017600169 is lying between 0.0029711647393913 to 0.0030918999126948 of Right as well as Left spread.
- The expected system length at Node-2, 0.000004164127 is lying between 0.0000039470168122 to 0.0000043207973729 of Right as well as Left spread in Trapezoidal fuzzy using LR Method, similarly expected system wating time at same node, 0.008075894209 is lying between 0.0077469722449914 to 0.0080952524169697 of Right as well as Left spread.
- The expected system length at Central Server (LHO), 0.000012581627 is lying between 0.0000123849415440 to 0.0000127579919797 of Right as well as Left spread in Trapezoidal fuzzy using LR Method, similarly expected system wating time at Central Server (LHO), 0.016775572796 is lying between 0.0165099989579280 to 0.0166781761653547 of Right as well as Left spread.

3.3. Comparative Behaviour of Node-1, Node-2, and Central Server

- The comparative analysis of Node-1, Node-2, and the Central Server under both triangular and trapezoidal fuzzy parameters shows a consistent pattern of queueing behaviour.
- In the triangular fuzzy model, Node-1 shows the lowest congestion, Node-2 shows moderate congestion, and the Central Server shows the highest congestion.
- For Node-1, the expected system length decreases from 3.7707×10^{-6} at $\alpha = 0$ to 9.2839×10^{-7} at $\alpha = 1$, while the mean waiting time decreases from 6.3077×10^{-3} to 2.9708×10^{-3}.
- For Node-2, the expected system length decreases from 1.0080×10^{-5} to 4.0994×10^{-6}, while the mean waiting time decreases from 1.2775×10^{-2} to 7.9504×10^{-3}.
- The Central Server records the highest values, with expected system length decreasing from 2.3426×10^{-5} to 1.2381×10^{-5}, and mean waiting time decreasing from 2.3115×10^{-2} to 1.6508×10^{-2}.

- Thus, under triangular fuzzy parameters, the congestion pattern is Node-1 < Node-2 < Central Server.
- A similar pattern is observed under trapezoidal fuzzy parameters. Node-1 remains the least congested, with expected system length decreasing from 1.0408 x 10^{-6} to 9.2868 x 10^{-7} and mean waiting time decreasing from 3.0919 x 10^{-3} to 2.9712 x 10^{-3}.
- Node-2 shows moderate congestion, with expected system length decreasing from 4.3208 x 10^{-6} to 3.9470 x 10^{-6}, and mean waiting time decreasing from 8.0953 x 10^{-3} to 7.7470 x 10^{-3}.
- The Central Server again shows the highest congestion, with expected system length decreasing from 1.2758 x 10^{-5} to 1.2385 x 10^{-5}, and mean waiting time decreasing from 1.6675 x 10^{-2} to 1.6510 x 10^{-2}.
- Therefore, under trapezoidal fuzzy parameters also, the congestion pattern remains Node-1 < Node-2 < Central Server.

4. Discussion

4.1. Queue Length Behaviour

The queue length results show that the expected system length decreases as the α -cut level increases from 0 to 1. This indicates that uncertainty reduces at higher α -cut levels and the queue length values become more stable.

In both triangular and trapezoidal fuzzy models, Node-1 has the lowest expected system length, Node-2 has moderate queue length, and the Central Server has the highest queue length. This is because the Central Server receives both direct arrivals and routed customers from Node-1 and Node-2.

Thus, the queue length pattern is: **Node-1<Node-2<Central Server**

The triangular fuzzy model has larger uncertainty ranges and the trapezoidal fuzzy model has smaller, steadier queue lengths. Overall, the results confirm that the Central Server is the main congestion point in the network.

4.2. Waiting Time Behaviour

As seen in the results of the waiting time, with increasing the α -cut level from 0 to 1, the mean waiting time is decreasing. This shows that the uncertainty decreases with increased α -cut levels and the system tends towards more stable performance values. In both triangular and trapezoidal fuzzy models, the mean waiting time for Node-1 is the lowest, the mean waiting time for Node-2 is moderate and the mean waiting time for Central Server is the highest. This is because the Central Server receives both direct customers and routed customers from both Node-1 as well as Node-2, thus increasing its service burden and delay.

Thus, the waiting time pattern is: **Node-1<Node-2<Central Server**

The triangular fuzzy model results in an uncertainty of waiting time with a larger range, the trapezoidal fuzzy model results in a more concentrated and more stable range of waiting time. The results overall, show that the Central Server is the most significant delay component of the network and that an increase in its service rate can be helpful in reducing the overall waiting time.

4.3. α-cut Behaviour

The α-cut level is significant in the understanding of the behaviour of the fuzzy queueing system. In this study, the values of the α-cuts are varied from 0 to 1 for triangular and trapezoidal fuzzy parameters. It is seen that at low values of α-cut, the bounds on arrival and service rates are wide resulting in wider intervals for expected queue length and mean waiting time. This means that the transient performance of the Node-1, Node-2 and Central Server are now more uncertain. The width of the fuzzy intervals and the corresponding LR spreads is decreasing with the increase in the level of α-cut. From the observed results, it can be seen that the left spread and right spread decreases with increasing value of α.

This behaviour is clearly seen in the LR-based results of expected system length and waiting time for all three service locations. With α=0 the uncertainty range is rather large since the fuzzy parameters are represented by the broadest interval. The closer α is to 1, the closer the interval will be to the mode or central value, and the smaller the spreads will be or even null. It indicates that the system performances are more accurate as the α-cuts are higher. The narrowing of spreads is important because it means a decrease in the uncertainty and a convergence of the fuzzy performance measures to crisp values. From a queueing perspective, it implies the estimated expected queue length and mean waiting time are more stable and reliable, the greater the value of α. Therefore the LR method aids in clearly identifying the decrease of uncertainty across the fuzzy system. The comparatively higher value of queue length and waiting time are still maintained in the Central Server due to the higher number of arrivals received in the server from direct and routed arrivals, but uncertainty is also reduced with the increase of α-cut levels. Hence, the behaviour of the α-cut validates the effectiveness of the proposed LR-based fuzzy Jackson network approach to model the uncertainty at lower values of α and to converge towards the definite queueing performance values at higher values of α.

4.3. Comparison between Triangular and Trapezoidal Fuzzy Models

In the present study, two types of fuzzy models namely triangular and trapezoidal fuzzy models are taken to investigate the impact of various types of fuzzy models on transient performance of the M/M/1 Jackson type of network queueing system. The expected length of the system and mean waiting time were numerically calculated at the Node-1, Node-2 and at the Central Server for both models when the LR representation is done based on α-cut. The results show that both triangular and trapezoidal fuzzy models follow a similar overall pattern, where the left and right spreads decrease as the α-cut level increases. This indicates that uncertainty gradually reduces and the fuzzy performance measures move towards more precise values. Generally the triangular fuzzy model yielded wider uncertainty bands compared with the trapezoidal fuzzy model. This is because a triangular fuzzy number expresses uncertainty about a single most likely value, the lower and upper fuzzy numbers are wider at lower levels of α-cut. In the numerical results, the triangular fuzzy tables show comparatively larger spreads for expected queue length and waiting time, especially at lower α values. This suggests a wider range of potential behaviours in the system for uncertain arrival and service rates. Thus, the use of triangular fuzzy modelling is suitable in cases where only a limited information about system parameters is available and only a single central or most probable value can be assumed. However, the trapezoidal fuzzy model resulted in more narrow and stable intervals. This is because trapezoidal fuzzy numbers have an interval as a core interval and not a modal value. This means the uncertainty is spread across a consistent set of plausible ranges of the parameters, leading to reduced LR spreads in performance measures.

In the numerical tables, the trapezoidal fuzzy results show more compact spreads for Node-1, Node-2, and the Central Server when compared with the corresponding triangular fuzzy results. This suggests that trapezoidal fuzzy modelling may be more appropriate when arrival and service rates are uncertain but are known to lie within a reliable core interval. The choice of fuzzy representation has a direct effect on queue performance interpretation. The width of the spreads in the triangular model show a higher level of uncertainty in the expected queue length and mean waiting time, while the trapezoidal model presents narrower spreads with a higher level of stability and good estimates. It can be seen however, that both the fuzzy models agree onNode-1 having the least congestion, Node-2 having moderate congestion and the Central server having the highest congestion level in terms of the queue length and waiting time, because of both direct and routed arrivals. Hence, the LR-based approach can be used to compare the effect of various fuzzy structure of the parameters on the transient performance of the queueing systems and reveal which service location in the network is most prone to congestion. The LR method helps to present the fuzzy queueing results in a simple and meaningful form. In the observed results, the centre value represents the main estimate of expected queue length and waiting time, while the left and right spreads show the uncertainty around that estimate. As the α-cut level increases, the spreads gradually become narrower, indicating reduction in uncertainty and movement towards crisp values. The triangular fuzzy model shows wider spreads, whereas the trapezoidal fuzzy model gives narrower and more stable spreads.

4.4. Quantitative Analysis of Uncertainty Reduction

To quantify uncertainty reduction, the percentage reduction in LR spread was calculated using:

$$Percent\ Reduction = \frac{Spread\ at\ \alpha = 0\ -\ Spread\ at\ \alpha = 1}{Spread\ at\ \alpha = 0}\ x\ 100$$

Since the left and right spreads are almost identical in the presented LR tables, the same percentage reduction applies to both left and right spreads. The values are based on the α-cut LR results reported in the triangular and trapezoidal fuzzy tables of the manuscript.

A. Triangular Fuzzy Model: Percentage Reduction in LR Spreads

In the triangular fuzzy model, the spreads decrease continuously from α = 0 to α = 1. At α = 1, the spreads become zero for all performance measures. Therefore, the uncertainty reduction is **100%** for Node-1, Node-2, and Central Server.

Table 4.4 (A) : Percentage Reduction in LR Spreads

Performance measure (Left and Right)	Spread at α = 0	Spread at α = 1	Percentage reduction
LN1 (Left and Right)	3.65624E-06	0.000000	100.00%
WN1 (Left and Right)	0.005575353	0.000000	100.00%
LN2 (Left and Right)	8.88128E-06	0.000000	100.00%
WN2 (Left and Right)	0.009288834	0.000000	100.00%
LHO (Left and Right)	1.82699E-05	0.000000	100.00%
WHO (Left and Right)	0.013948479	0.000000	100.00%

B. Trapezoidal Fuzzy Model: Percentage Reduction in LR Spreads

In the trapezoidal fuzzy model, the spreads also decrease as α increases, but they do not become zero at α = 1 because trapezoidal fuzzy numbers retain a core interval.

Table 4.4 (B) : Percentage Reduction in LR Spreads

Performance measure (Left and Right)	Spread at α = 0	Spread at α = 1	Percentage reduction
LN1 (Left and Right)	5.66399E-07	2.81114E-08	95.04%
WN1 (Left and Right)	0.001194098	6.02448E-05	94.95%
LN2 (Left and Right)	1.65672E-06	3.19267E-07	80.73%
WN2 (Left and Right)	0.002133672	0.000418175	80.40%
LHO (Left and Right)	3.74502E-06	3.73172E-07	90.04%
WHO (Left and Right)	0.003325584	0.000332467	90.00%

The quantitative results confirm that uncertainty decreases substantially as the α-cut level increases. The LR spreads for the triangular model decrease by 100%, converging totally to crisp values at $\alpha = 1$. The uncertainty reduction in the trapezoidal fuzzy model is between 80.40% and 95.04%, which shows strong uncertainty reduction but not complete since trapezoidal fuzzy numbers allow for a range of core uncertainty. The trapezoidal results are observed that the uncertainty reduction of Node-1 is highest while the uncertainty reduction of Node-2 is relatively lower. In general, the results show that the more the value of α is increased, the more the uncertainty intervals are narrowed and the more stable the estimates of queue length and waiting time are.

4.4. Practical Significance of the Findings

- The results of the current study will be important in practice for the identification of bottlenecks in coupled service systems. Under both triangular and trapezoidal fuzzy models, the numerical results demonstrate that the expected length of queues and mean waiting time of the Central Server are higher than the expected length of queues and mean waiting time of Node-1 and Node-2. This means that the service capacity of the Central Server will need to be increased, or the processing speed improved, or more service facilities will be needed to alleviate congestion.
- The waiting time at the Central Server was found to be larger than that at other servers indicating that if the service rate at the Central Server is increased, it would be beneficial for the network as the overall waiting time would decrease. The customers arriving at the Central Server are either direct or routed customers from Node-1 and Node-2, so any increase in the efficiency of the service at the Central Server will have a beneficial effect on the whole queueing network.
- The results show also that the routing probabilities are an important factor in congestion at the Central Server. As the number of customers that are transferred from Node-1 and Node-2 to the Central Server grows, the workload on the central service point grows. Hence, suitable routing control is required to maintain a balance between the customer traffic to prevent overloading the Central Server.
- The LR representation offers a practical support to the decision making, indicating the main performance value and the uncertainty associated with it with the left and right spreads. The spread reflects uncertainty and variability in system performance, with wider spreads signifying more uncertainty and greater variability of system performance and narrower spreads signifying more stable behaviour. Hence, the LR method can be used to identify areas of uncertainty and areas where operational improvement is most required by the decision makers..

5. Conclusion

In this study, the transient analysis of an M/M/1 Jackson type queueing network with fuzzy arrival and service parameters was proposed using the LR method. The model consists of two service nodes and one Central Server, with finite capacity, routing and task abandonment. The transient probabilities were obtained by numerical solution by the fourth order Runge – Kutta method and the triangular and trapezoidal fuzzy parameters were represented by the centre value and left – right spread.

Based on the numerical results it was observed that the proposed LR method give a compact and meaningful representation of the fuzzy queueing performance measures. In both triangular and trapezoidal fuzzy models, the congestion of the system length and mean waiting time exhibited the same relative order: the Central Server was the most congested, followed by the moderate level of congestion in Node-2, and finally having the least level of congestion in Node-1. This is primarily because of the mix of customers directly arriving and the routed customers from Node-1 and Node-2 in the Central Server.

The results also showed that the uncertainty is reduced when the α-cut level is increased. In the triangular fuzzy model, the LR spreads converged to zero at $\alpha = 1$, which meant that the LR spreads converged to crisp value. The spreads in the trapezoidal fuzzy model also dropped significantly, and there was still some uncertainty as a result of the core interval. The trapezoidal fuzzy model resulted in smaller and more consistent uncertainty intervals than the triangular fuzzy model..

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AI Declaration: The authors use AI-assisted language support only for grammar checking, sentence clarity, and formatting improvement. The scientific content, statistical analysis, interpretation, results, references and for the final manuscript were fully considered and approved by the authors. No AI tool was used to generate data that fabricate results or replace author responsibility.

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Appendix 1 : Derived Kolmogorov forward differential equations

The Transient state equations governing the various probabilities are detailed in the form of following Differential equations:

1. $\frac{dp_{(0.0.0)}^{(t)}}{dt} = -(\lambda_x b_1 + \lambda_y b_2 + \lambda_h b_3)\ p_{(0.0.0)}^{(t)} + \mu_x(1-\mathrm{p}_1)\, p_{(1.0.0)}^{(t)} + \mu_y(1-\mathrm{p}_2)\, p_{(0.1.0)}^{(t)} + \mu_h\, p_{(0.0.1)}^{(t)}$

2. $\frac{dp_{(j.0.0)}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_x + (\mathrm{j}-1)\xi_1) + \lambda_y b_2 + \lambda_h b_3)\mathrm{P}_{(\mathrm{j},0,0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{j}-1,0,0)}^{(t)} + \mu_x(1-\mathrm{p}_1)\mathrm{P}_{(\mathrm{j}+1,0,0)}^{(t)} + \mu_y(1-\mathrm{p}_2)\mathrm{P}_{(\mathrm{j},1,0)}^{(t)} + \mu_h \mathrm{P}_{(\mathrm{j},0,1)}^{(t)}\ ;\ 1 \le j \le s-1$

3. $\frac{dP_{(\mathrm{s},0,0)}^{(t)}}{dt} = -(\lambda_y b_2 + (\mu_x + (\mathrm{s}-1)\xi_1) + \lambda_h b_3)\mathrm{P}_{(\mathrm{s},0,0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{s}-1,0,0)}^{(t)} + \mu_y(1-\mathrm{p}_2)\mathrm{P}_{(\mathrm{s},1,0)}^{(t)} + \mu_h \mathrm{P}_{(\mathrm{s},0,1)}^{(t)}$

4. $\frac{dP_{(0,\mathrm{k},0)}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_y + (k-1)\xi_2) + \lambda_y b_2 + \lambda_h b_3)\mathrm{P}_{(0,\mathrm{k},0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(0,\mathrm{k}-1,0)}^{(t)} + (\mu_y(1-\mathrm{p}_2) + \mathrm{k}\xi_2)\mathrm{P}_{(0,\mathrm{k}+1,0)}^{(t)} + \mu_x(1-\mathrm{p}_1)\mathrm{P}_{(1,\mathrm{k},0)}^{(t)} + \mu_h \mathrm{P}_{(0,\mathrm{k},1)}^{(t)};\ 1 \le k \le s-1$

5. $\frac{dP_{(0,\mathrm{s},0)}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_y + (s-1)\xi_2) + \lambda_h b_3)\mathrm{P}_{(0,\mathrm{s},0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(0,\mathrm{s},0)}^{(t)} + \mu_x(1-\mathrm{p}_1)\mathrm{P}_{(1,\mathrm{s},0)}^{(t)} + \mu_h \mathrm{P}_{(0,\mathrm{s},1)}^{(t)}$

6. $\frac{dP_{(0,0,\mathrm{h})}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_h + (\mathrm{h}-1)\xi_3 + \lambda_y b_2 + \lambda_h b_3)\mathrm{P}_{(0,0,\mathrm{h})}^{(t)} + \lambda_h b_3 \mathrm{P}_{(0,\mathrm{s},\mathrm{h}-1)}^{(t)} + (\mu_h + \mathrm{h}\xi_3))\mathrm{P}_{(0,0,\mathrm{h}+1)}^{(t)} + \mu_x \mathrm{p}_1 \mathrm{P}_{(1,0,\mathrm{h}-1)}^{(t)} + \mu_y \mathrm{p}_2 \mathrm{P}_{(0,1,\mathrm{h}-1)}^{(t)}\ ;\ 1 \le h \le s-1$

7. $\frac{dP_{(0,0,\mathrm{s})}^{(t)}}{dt} = -(\lambda_x b_1 + \lambda_y b_2 + (\mu_h + (\mathrm{s}-1)\xi_3)\mathrm{P}_{(0,0,\mathrm{s})}^{(t)} + \lambda_h b_3 \mathrm{P}_{(0,0,\mathrm{s}-1)}^{(t)} + \mu_x \mathrm{p}_1 \mathrm{P}_{(1,0,\mathrm{s}-1)}^{(t)} + \mu_y \mathrm{p}_2 \mathrm{P}_{(0,1,\mathrm{s}-1)}^{(t)}$

8. $\frac{dP_{(\mathrm{j},\mathrm{k},0)}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_x + (\mathrm{j}-1)\xi_1) + \lambda_y b_2 + (\mu_y + (\mathrm{k}-1)\xi_2) + \lambda_h b_3)\mathrm{P}_{(\mathrm{j},\mathrm{k},0)}^{(t)} + (\mu_x(1-\mathrm{p}_1) + \mathrm{j}\xi_1)\mathrm{P}_{(\mathrm{j}+1,\mathrm{k},0)}^{(t)} + (\mu_y(1-\mathrm{p}_2) + \mathrm{k}\xi_2)\mathrm{P}_{(\mathrm{j},\mathrm{k}+1,0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{j}-1,\mathrm{k},0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(\mathrm{j},\mathrm{k}-1,0)}^{(t)} + +\mu_h \mathrm{P}_{(\mathrm{j},\mathrm{k},1)}^{(t)};\ 1 \le j \le s-1, 1 \le k \le s-1$

9. $\frac{dP_{(\mathrm{j},\mathrm{s},0)}^{(t)}}{dt} = -(\lambda_x b_1 + (\mu_x + (\mathrm{j}-1)\xi_1) + (\mu_y + (\mathrm{s}-1)\xi_2) + \lambda_h b_3)\mathrm{P}_{(\mathrm{j},\mathrm{s},0)}^{(t)} + (\mu_x(1-\mathrm{p}_1) + \mathrm{j}\xi_1)\mathrm{P}_{(\mathrm{j}+1,\mathrm{s},0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{j}-1,\mathrm{s},0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(\mathrm{j},\mathrm{s}-1,0)}^{(t)} + \mu_h \mathrm{P}_{(\mathrm{j},\mathrm{s},1)}^{(t)}\ ;\ 1 \le j \le s-1$

10. $\frac{dP_{(\mathrm{s},\mathrm{k},0)}^{(t)}}{dt} = -((\mu_x + (s-1)\xi_1) + \lambda_y b_2 + (\mu_y + (\mathrm{k}-1)\xi_2) + \lambda_h b_3)\mathrm{P}_{(\mathrm{s},\mathrm{k},0)}^{(t)} + (\mu_y(1-\mathrm{p}_2) + \mathrm{k}\xi_2)\mathrm{P}_{(\mathrm{s},\mathrm{k}+1,0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(\mathrm{s},\mathrm{k}-1,0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{s}-1,\mathrm{k},0)}^{(t)} + \mu_h \mathrm{P}_{(\mathrm{s},\mathrm{k},1)}^{(t)};\ 1 \le k \le s-1$

11. $\frac{dP_{(\mathrm{s},\mathrm{s},0)}^{(t)}}{dt} = -((\mu_x + (s-1)\xi_1) + (\mu_y + (\mathrm{s}-1)\ \xi_2) + \lambda_h b_3)\mathrm{P}_{(\mathrm{s},\mathrm{s},0)}^{(t)} + \lambda_x b_1 \mathrm{P}_{(\mathrm{s}-1,\mathrm{s},0)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(\mathrm{s},\mathrm{s}-1,0)}^{(t)}$

12. $\frac{dP_{(0,\mathrm{k},\mathrm{h})}^{(t)}}{dt} = -(\lambda_x b_1 + \lambda_y b_2 + (\mu_y + (\mathrm{k}-1)\xi_2) + \lambda_h b_3 + (\mu_h + (h-1)\xi_3))\mathrm{P}_{(0,\mathrm{k},\mathrm{h})}^{(t)} + \mu_x(1-p_1)\mathrm{P}_{(1,\mathrm{k},\mathrm{h})}^{(t)} + \mu_y(1-p_2)\mathrm{P}_{(0,\mathrm{k}+1,\mathrm{h})}^{(t)} + (\mu_h + \mathrm{h}\xi_3)\mathrm{P}_{(0,\mathrm{k},\mathrm{h}+1)}^{(t)} + \lambda_h b_3 \mathrm{P}_{(0,\mathrm{k},\mathrm{h}-1)}^{(t)} + \lambda_y b_2 \mathrm{P}_{(0,\mathrm{k}-1,\mathrm{h})}^{(t)};1 \le k \le s-1, 1 \le h \le s-1$

13. $\frac{dP_{(0,s,h)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_y + (s-1)\xi_2) + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(0,s,h)}^{(t)} + (\mu_h + h\xi_3)P_{(0,s,h+1)}^{(t)} + \lambda_h b_3 P_{(0,s,h-1)}^{(t)} + \lambda_y b_2 P_{(0,s,h)}^{(t)} + \mu_x(1-p_1)P_{(1,s,h)}^{(t)}; 1 \le h \le s-1$

14. $\frac{dP_{(0,k,s)}^{(t)}}{dt} = -\big(\lambda_x b_1 + \lambda_y b_2 + (\mu_y + (k-1)\xi_2) + (\mu_h + (s-1)\xi_3)\big)P_{(0,k,s)}^{(t)} + (\mu_y(1-p_2) + \mathrm{k}\xi_2)P_{(0,k+1,s)}^{(t)} + \lambda_h b_3 P_{(0,k,s-1)}^{(t)} + \lambda_y b_2 P_{(0,k-1,s)}^{(t)} + \mu_x(1-p_1)P_{(1,s,h)}^{(t)}; 1 \le k \le s-1$

15. $\frac{dP_{(0,s,s)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_y + (s-1)\xi_2) + (\mu_h + (s-1)\xi_3)\big)P_{(0,s,s)}^{(t)} + \lambda_h b_3 P_{(0,s,s-1)}^{(t)} + \lambda_y b_2 P_{(0,s-1,s)}^{(t)} + \mu_x(1-p_1)P_{(1,s,s)}^{(t)}; 1 \le k \le s-1$

16. $\frac{dP_{(j,0,h)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_x + (j-1)\xi_1) + \lambda_y b_2 + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(j,0,h)}^{(t)} + (\mu_x(1-p_1) + \mathrm{j}\xi_1)P_{(j+1,0,h)}^{(t)} + (\mu_h + h\xi_3)P_{(j,0,h+1)}^{(t)} + \lambda_x b_1 P_{(j-1,0,h)}^{(t)}\mathrm{P(j-1,0,h)} + \lambda_h b_3 P_{(j,0,h-1)}^{(t)} + \mu_y(1-p_2)P_{(j,1,h)}^{(t)}; 1 \le j \le s-1, 1 \le h \le s-1$

17. $\frac{dP_{(s,0,h)}^{(t)}}{dt} = -\big((\mu_x + (s-1)\xi_1) + \lambda_y b_2 + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(s,0,h)}^{(t)} + \lambda_x b_1 P_{(s-1,0,h)}^{(t)} + (\mu_h + h\xi_3)P_{(s,0,h+1)}^{(t)} + \lambda_h b_3 P_{(s,0,h-1)}^{(t)} + \mu_y(1-p_2)P_{(s,1,h)}^{(t)}; 1 \le h \le s-1$

18. $\frac{dP_{(j,0,s)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_x + (j-1)\xi_1) + \lambda_y b_2 + (\mu_h + (s-1)\xi_3)\big)P_{(j,0,s)}^{(t)} + \lambda_h b_3 P_{(j,0,s-1)}^{(t)} + (\mu_x(1-p_1) + \mathrm{j}\xi_1)P_{(j+1,0,s)}^{(t)} + \lambda_x b_1 P_{(j-1,0,s)}^{(t)} + \mu_y(1-p_2)P_{(j,1,s)}^{(t)}; 1 \le j \le s-1$

19. $\frac{dP_{(s,0,s)}^{(t)}}{dt} = -\big((\mu_x + (s-1)\xi_1) + \lambda_y b_2 + (\mu_h + (s-1)\xi_3)\big)P_{(s,0,s)}^{(t)} + \lambda_x b_1 P_{(s,0,s)}^{(t)} + \lambda_h b_3 P_{(s,0,s-1)}^{(t)} + \mu_y(1-p_2)P_{(s,1,s)}^{(t)}$

20. $\frac{dP_{(j,k,h)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_x + (j-1)\xi_1) + \lambda_y b_2 + (\mu_y + (k-1)\xi_2) + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(j,k,h)}^{(t)} + (\mu_x(1-p_1) + (\mathrm{j})\xi_1)P_{(j+1,k,h)}^{(t)} + (\mu_y(1-p_2) + k\xi_2)P_{(j,k+1,h)}^{(t)} + (\mu_h + (h)\xi_3)P_{(j,k,h+1)}^{(t)} + \lambda_x b_1 P_{(j-1,k,h)}^{(t)} + \lambda_y b_2 P_{(j,k-1,h)}^{(t)} + \lambda_h b_3 P_{(j,k,h-1)}^{(t)}; 1 \le j \le s-1, 1 \le k \le s-1, 1 \le h \le s-1$

21. $\frac{dP_{(s,k,h)}^{(t)}}{dt} = -\big((\mu_x + (s-1)\xi_1) + \lambda_y b_2 + (\mu_y + (k-1)\xi_2) + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(s,k,h)}^{(t)} + \big(\mu_y(1-p_2) + k\xi_2\big)P_{(s+1,k+1,h)}^{(t)} + (\mu_h + h\xi_3)P_{(s+1,k,h+1)}^{(t)} + \lambda_x b_1 P_{(s-1,k,h)}^{(t)} + \lambda_y b_2 P_{(s,k-1,h)}^{(t)} + \lambda_h b_3 P_{(s,k,h-1)}^{(t)}; 1 \le k \le s-1, 1 \le h \le s-1$

22. $\frac{dP_{(j,s,h)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_x + (j-1)\xi_1) + (\mu_y + (s)\xi_2) + \lambda_h b_3 + (\mu_h + (h-1)\xi_3)\big)P_{(j,s+1,h)}^{(t)} + (\mu_x(1-p_1) + j\xi_1)P_{(j+1,s,h)}^{(t)} + (\mu_h + h\xi_3)P_{(j,s,h+1)}^{(t)} + \lambda_x b_1 P_{(j-1,s,h)}^{(t)} + \lambda_y b_2 P_{(j,s,h)}^{(t)} + \lambda_h b_3 P_{(j,s,h-1)}^{(t)}; 1 \le j \le s-1, 1 \le h \le s-1$

23. $\frac{dP_{(j,k,s)}^{(t)}}{dt} = -\big(\lambda_x b_1 + (\mu_x + (j-1)\xi_1) + \lambda_y b_2 + (\mu_y + (k-1)\xi_2) + (\mu_h + (s-1)\xi_3)\big)P_{(j,k,s)}^{(t)} + (\mu_x(1-p_1) + j\xi_1)P_{(j+1,k,s)}^{(t)} + (\mu_y(1-p_2) + k\xi_2)P_{(j,k+1,s)}^{(t)} + \lambda_x b_1 P_{(j-1,k,s)}^{(t)} + \lambda_y b_2 P_{(j,k-1,s)}^{(t)} + \lambda_h b_3 P_{(j,k,s-1)}^{(t)}; 1 \le j \le s-1, 1 \le k \le s-1$

24. $\frac{dP_{(s,s,s)}^{(t)}}{dt} = -\big((\mu_x + (s-1)\xi_1) + (\mu_y + (s-1)\xi_2) + (\mu_h + (s-1)\xi_3)\big)P_{(s,s,s)}^{(t)} + \lambda_x b_1 P_{(s-1,s,s)}^{(t)} + \lambda_y b_2 P_{(s,s-1,s)}^{(t)} + \lambda_h b_3 P_{(s,s,s-1)}^{(t)}$